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Sobolev space theory of SPDEs with continuous or measurable leading coefficients

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Abstract

We study stochastic partial differential equations with variable coefficients defined on $\mathbb{R}^d, \mathbb{R}^d_+$ and bounded C^1 domains. For equations with continuous leading coefficients we give existence and uniqueness results in $L_q(L_p)$-spaces, where it is allowed for the powers of summability with respect to space and time variables to be different. For equations with measurable leading coefficients we give unique solvability in L_p-spaces.

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1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\{\mathcal{F}_t, t \geq 0\}$ be an increasing filtration of σ-fields $\mathcal{F}_t \subset \mathcal{F}$, each of which contains all $(\mathcal{F}, P)$-null sets. We assume that on Ω we are given independent one-dimensional Wiener processes $w^1_t, w^2_t, \ldots$, each of which is a Wiener process relative to $\{\mathcal{F}_t, t \geq 0\}$.

In this article we are dealing with the Sobolev space theory of the equations

$$\mathrm{d}u = (a^{ij}u_{x^ix^j} + b^iu_{x^i} + cu + f)\mathrm{d}t + (\sigma^{ik}u_{x^i} + \nu^k u + g^k)\mathrm{d}w^k_t, \tag{1.1}$$

$$\mathrm{d}u = (D_i(a^{ij}u_{x^j} + \bar{b}^iu + f^i) + b^iu_{x^i} + cu + f)\mathrm{d}t + (\sigma^{ik}u_{x^i} + \nu^k u + g^k)\mathrm{d}w^k_t \tag{1.2}$$

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given for $t \geq 0$ and $x \in \mathcal{O} \subset \mathbb{R}^d$. Here i and j go from 1 to d, and k runs through $\{1, 2, \ldots\}$ with the summation convention being enforced.

The first goal of this article is to extend the results in [7], where an $L_q(L_p)$-theory is constructed for the equation

$$\mathrm{d}u = (a^{ij} u_{x^i x^j} + f)\mathrm{d}t + (\sigma^{ik} u_{x^i} + g^k)\mathrm{d}w_t^k \tag{1.3}$$

with constant coefficients in $\mathbb{R}^d$ and $\mathbb{R}^d_+$. We give the unique solvability of Eqs. (1.1) and (1.2) with continuous leading coefficients in (weighted) $L_q(L_p)$-spaces defined on C^1 domains. The number of derivatives of the solutions can be any real number, in particular, it can be negative and fractional. Our results are new even for Eq. (1.3) with constant coefficients since ours are obtained for the wider range of weights (see Lemma 5.1). Once one knows how to solve Eq. (1.3) with constant coefficients in $\mathbb{R}^d$ and $\mathbb{R}^d_+$, then one may regard constructing a solvability theory of Eq. (1.1) with variable coefficients as a standard work. If $p \neq q$ then $L_q(L_p)$-theory of SPDEs turns out to be an exception to the usual situation. For instance, relations like

$$E \int_0^T \|u(t,\cdot)\|^q_{L_p(\mathbb{R}^d)} \mathrm{d}t \sim \sum_{n=1}^{\infty} E \int_0^T \|\zeta_n(\cdot) u(t,\cdot)\|^q_{L_p(\mathbb{R}^d)} \mathrm{d}t$$

hold only if $p = q$, where $\{\zeta_n : n = 1, 2, \ldots\}$ is a standard partition of unity of $\mathbb{R}^d$. Thus local estimations of u, which may be obtained by a perturbation argument, don't easily yield a priori estimate of u. Furthermore, since we are also dealing with the equations in Sobolev spaces with weights, usual perturbation arguments don't work well and require some nontrivial modifications.

We refer to [2,3,8,10,11] and references therein for the L_p-theory of SPDEs with continuous leading coefficients. We also refer to [4] for the weighted L_p-theory of elliptic and parabolic PDEs in C^1 domains. Many advantages of $L_q(L_p)$-theory over L_p-theory are investigated in [6]. We introduce some of them in Corollary 2.18.

The second goal of this article is to extend the results of [1] and to present an L_p-theory of Eq. (1.2) with measurable coefficients. Let $\rho(x) = \mathrm{dist}(x, \partial\mathcal{O})$, $L_{p,\theta}(\mathcal{O}) = L_p(\mathcal{O}, \rho^{\theta-d}(x)\mathrm{d}x)$ and

$$\mathbb{L}_{p,\theta}(\mathcal{O}, T) = L_p(\Omega \times [0, T], L_{p,\theta}(\mathcal{O})),$$
$$\mathbb{H}^1_{p,\theta-p}(\mathcal{O}, T) = \{u : \rho^{-1}u, u_x \in \mathbb{L}_{p,\theta}(\mathcal{O}, T)\}.$$

In [1], the unique solvability of Eq. (1.2) is constructed in $\mathbb{H}^1_{p,\theta-p}(\mathcal{O}, T)$ when $\sigma^{ik} = 0$ and $\theta \approx d$. In this article the results in [1] are extended as follows. First, the condition $\sigma^{ik} = 0$ is dropped. Second, the condition $\theta \approx d$ is relaxed and replaced by

$$(d - \kappa_1) \wedge \left(d + p - 2 - \frac{1}{\frac{Kp}{\delta_0(p-1)} - 1}\right) < \theta < d + p - 2 + \frac{1}{\frac{Kp}{\delta_0(p-1)} + 1},$$

where the constants κ_1, δ_0 and K are specified in Section 2.4. This extension of the range of θ contributes to the Hölder estimation of the solutions (see Corollary 2.18).

We also remark that in this paper we are allowing our coefficients to be unbounded and to blow up near the boundary of $\mathcal{O}$ (see Remark 2.14).

As usual $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1, \ldots, x^d)$, $B_r(x) = \{y \in \mathbb{R}^d : |x - y| < r\}$, $B_r = B_r(0)$ and $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x^1 > 0\}$. For $i = 1, \ldots, d$, multi-indices

$\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \{0, 1, 2, \ldots\}$, and functions $u(x)$ we set

$$u_{x^i} = \partial u/\partial x^i = D_i u, \quad D^\alpha u = D_1^{\alpha_1} \cdot \cdots \cdot D_d^{\alpha_d} u, \quad |\alpha| = \alpha_1 + \cdots + \alpha_d.$$

If we write $N = N(\cdots)$, this means that the constant N depends only on what are in parenthesis. Throughout the article, for functions depending on ω, t and x, the argument $\omega \in \Omega$ will be omitted.

2. Main results

2.1. SPDEs with continuous leading coefficients on $\mathbb{R}^d$

In this section we present an $L_q(L_p)$-theory of Eqs. (1.1) and (1.2) defined on $\mathbb{R}^d$. For $n = 0, 1, 2, \ldots$ and $p \in (1, \infty)$ define

$$H_p^n = H_p^n(\mathbb{R}^d) = \{u : u, Du, \ldots, D^\alpha u \in L_p : |\alpha| \le n\}.$$

In general, for $\gamma \in \mathbb{R}$ we denote $H_p^\gamma = (1 - \Delta)^{-\gamma/2} L_p$ and define

$$\|u\|_{H_p^\gamma} := \|(1 - \Delta)^{\gamma/2} u\|_{L_p} < \infty.$$

This definition is also used for ℓ_2-valued functions $g = (g^1, g^2, \ldots)$, in which case,

$$\|g\|_{H_p^\gamma} = \|g\|_{H_p^\gamma(\ell_2)} = \||(1 - \Delta)^{\gamma/2} g|_{\ell_2}\|_{L_p}.$$

By $\mathcal{P}$ we denote the predictable σ-field generated by $\{\mathcal{F}_t, t \ge 0\}$. For the stopping time τ, denote $(\!(0, \tau]\!] = \{(\omega, t) : 0 < t \le \tau(\omega)\}$,

$$\mathbb{H}_p^{\gamma,q}(\tau) = L_q((\!(0, \tau]\!], \mathcal{P}, H_p^\gamma), \qquad U_p^{\gamma,q} = L_q(\Omega, \mathcal{F}_0, H_p^{\gamma - 2/q}),$$
$$\mathbb{L}_p^q(T) = \mathbb{H}_p^{0,q}(T).$$

We write $u \in \mathcal{H}_p^{\gamma,q}(\tau)$ if $u \in \mathbb{H}_p^{\gamma,q}(\tau)$, $u(0, \cdot) \in U_p^{\gamma,q}$ and for some $f \in \mathbb{H}_p^{\gamma-2,q}(\tau)$, $g = (g^1, g^2, \ldots) \in \mathbb{H}_p^{\gamma-1,q}(\tau)$

$$\mathrm{d}u = f \mathrm{d}t + g^k \mathrm{d}w_t^k$$

in the sense of distributions. In other words, for any $\phi \in C_0^\infty$, the equality

$$(u(t, \cdot), \phi) = (u(0), \phi) + \int_0^t (f(s, \cdot), \phi) \mathrm{d}s + \sum_{k=1}^\infty \int_0^t (g^k(s, \cdot), \phi) \mathrm{d}w_s^k$$

holds for all $t \le \tau$ with probability 1. Denote

$$\mathcal{H}_{p,0}^{\gamma,q}(\tau) = \mathcal{H}_p^{\gamma,q}(\tau) \cap \{u : u(0, \cdot) = 0\}.$$

The norm in $\mathcal{H}_p^{\gamma,q}(\tau)$ is introduced by

$$\|u\|_{\mathcal{H}_p^{\gamma,q}(\tau)} = \|u\|_{\mathbb{H}_p^{\gamma,q}(\tau)} + \|f\|_{\mathbb{H}_p^{\gamma-2,q}(\tau)} + \|g\|_{\mathbb{H}_p^{\gamma-1,q}(\tau)} + \|u_0\|_{U_p^{\gamma,q}}.$$

In the above notation we drop q if $p = q$.

Throughout the article we assume that for each x, the functions $a^{ij}(t, x)$, $b^i(t, x)$, $\bar{b}^i(t, x)$, $c(t, x)$, $\sigma^{ik}(t, x)$ and $\nu^k(t, x)$ are predictable functions of (ω, t). Also we assume that

$$2 \le p \le q < \infty, \quad \tau \le T.$$

By I we denote the $d \times d$ identity matrix. For $d \times d$ matrices $A = (a^{ij})$ and $B = (b^{ij})$ we write $A < B$ (resp, $A \leq B$) if for any $\xi \in \mathbb{R}^d$, $\xi \neq 0$,

$$a^{ij}\xi^i\xi^j < b^{ij}\xi^i\xi^j \quad (\text{resp}, a^{ij}\xi^i\xi^j \leq b^{ij}\xi^i\xi^j).$$

Assumption 2.1. (i) The leading coefficients a^{ij} and σ^i are uniformly continuous in x. In other words, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|a^{ij}(t,x) - a^{ij}(t,y)| + |\sigma^i(t,x) - \sigma^i(t,y)|_{\ell_2} < \varepsilon$$

whenever $|x-y| < \delta$.

(ii) There exists a constant $\delta \in (0,1)$ such that for any ω, t, x

$$\delta^{-1} I \geq (a^{ij}) \geq \delta I, \quad (1-\delta)(a^{ij}) \geq (\alpha^{ij}), \tag{2.1}$$

where $\alpha^{ij} := 1/2 \sum_k \sigma^{ik}\sigma^{jk}$.

Assumption 2.2. For each ω, t, x

$$|b^i(t,x)| + |\bar{b}^i(t,x)| + |c(t,x)| + |\nu(t,x)|_{\ell_2} \leq K.$$

Fix $\varepsilon_0 = \varepsilon_0(p,q)$ such that, $\varepsilon_0 = 0$ if $p = q$, and $\varepsilon_0 > 0$ otherwise.

Theorem 2.3. *Let Assumptions 2.1 and 2.2 be satisfied. Then for any $f^i \in \mathbb{L}^q_p(\tau)$, $f \in \mathbb{H}^{-1,q}_p(\tau)$, $g \in \mathbb{L}^q_p(\tau)$ and $u_0 \in L_q(\Omega, \mathcal{F}_0, H_p^{1-2/q+\varepsilon_0})$, Eq. (1.2) with initial data u_0 admits a unique solution u in the class $\mathcal{H}^{1,q}_p(\tau)$, and for this solution*

$$\|u\|^q_{\mathcal{H}^{1,q}_p(\tau)} \leq N\|f^i\|^q_{\mathbb{L}^q_p(\tau)} + N\|f\|^q_{\mathbb{H}^{-1,q}_p(\tau)} + N\|g\|^q_{\mathbb{L}^q_p(\tau)} + NE\|u_0\|^q_{H_p^{1-2/q+\varepsilon_0}}, \tag{2.2}$$

where $N = N(\varepsilon_0, d, p, q, \delta, K, T)$.

Fix $\kappa > 0$. For $\gamma \in \mathbb{R}$ define $|\gamma|_+ = |\gamma|$ if $|\gamma| = 0, 1, 2\ldots$ and $|\gamma|_+ = |\gamma| + \kappa$ otherwise. Also define

$$B^{|\gamma|_+} = \begin{cases} B(\mathbb{R}^d) & : \quad \gamma = 0 \\ C^{|\gamma|-1,1}(\mathbb{R}^d) & : \quad |\gamma| = 1, 2, \ldots \\ C^{|\gamma|+\kappa}(\mathbb{R}^d) & : \quad \text{otherwise,} \end{cases}$$

where B is the space of bounded functions, and $C^{|\gamma|-1,1}$, $C^{|\gamma|+\kappa}$ are usual Hölder spaces. We also use the Banach space $B^{|\gamma|_+}$ for ℓ_2-valued functions.

Assumption 2.4. For each ω and $t > 0$,

$$|a^{ij}(t,\cdot)|_{B^{|\gamma|_+}} + |b^i(t,\cdot)|_{B^{|\gamma|_+}} + |c(t,\cdot)|_{B^{|\gamma|_+}} + |\sigma^i(t,\cdot)|_{B^{|\gamma+1|_+}} + |\nu(t,\cdot)|_{B^{|\gamma+1|_+}} \leq C.$$

Theorem 2.5. *Let Assumptions 2.1, 2.2 and 2.4 be satisfied. Then for any $f \in \mathbb{H}^{\gamma,q}_p(\tau)$, $g \in \mathbb{H}^{\gamma+1,q}_p(\tau)$ and $u_0 \in L_q(\Omega, \mathcal{F}_0, H_p^{\gamma+2-2/q+\varepsilon_0})$, Eq. (1.1) with initial data u_0 has a unique solution $u \in \mathcal{H}^{\gamma+2,q}_p(\tau)$ and for this solution*

$$\|u\|^q_{\mathcal{H}^{\gamma+2,q}_p(\tau)} \leq N\|f\|^q_{\mathbb{H}^{\gamma,q}_p(\tau)} + N\|g\|^q_{\mathbb{H}^{\gamma+1,q}_p(\tau)} + NE\|u_0\|^q_{H_p^{\gamma+2-2/q+\varepsilon_0}}, \tag{2.3}$$

where $N = N(\gamma, \varepsilon_0, d, p, q, \delta, K, C, T)$.

Remark 2.6. One can get some Hölder estimates of the solutions by formally taking $\psi = 1$ in Corollary 2.18.

2.2. SPDEs with measurable coefficients on $\mathbb{R}^d$

In this section we present an L_p-theory of Eq. (1.2) with measurable coefficients on $\mathbb{R}^d$.

Assumption 2.7. There exist constants $\delta_0 \in (0, 1]$, $K \in [1, \infty)$ such that for any ω, t, x

$$\delta_0 I < (a^{ij} - \alpha^{ij}) \leq (a^{ij}) < KI. \tag{2.4}$$

Theorem 2.8. *Let Assumptions 2.2 and 2.7 hold and σ^i be uniformly continuous in x. Then there exists a constant $p_0 = p_0(\delta_0, K, d) > 2$ such that if $p \in [2, p_0)$ then for any $f^i \in \mathbb{L}_p(\tau)$, $g \in \mathbb{L}_p(\tau, \ell_2)$, $f \in \mathbb{H}_p^{-1}(\tau)$ and $u_0 \in U_p^1$ Eq. (1.2) admits a unique solution $u \in \mathcal{H}_p^1(\tau)$, and*

$$\|u\|_{\mathcal{H}_p^1(\tau)} \leq N(\|f^i\|_{\mathbb{L}_p(\tau)} + \|f\|_{\mathbb{H}_p^{-1}(\tau)} + \|g\|_{\mathbb{L}_p(\tau)} + \|u_0\|_{U_p^1}), \tag{2.5}$$

where the constant N is independent of f^i, f, g, u_0 and u.

The continuity of σ is dropped in the following result.

Theorem 2.9. *Let Assumption 2.2 hold true. Suppose that there exists a constant $\delta_1 \in (0, \infty)$ such that*

$$(\alpha^{ij}) < \delta_1 I < (\delta_0 + \delta_1)I < (a^{ij}) < KI. \tag{2.6}$$

Then there exists $p_1 = p_1(\delta_i, K) > 2$ such if $p \in [2, p_1)$ then for any $f^i \in \mathbb{L}_p(\tau)$, $g \in \mathbb{L}_p(\tau, \ell_2)$, $f \in \mathbb{H}_p^{-1}(\tau)$ and $u_0 \in U_p^1$ Eq. (1.2) has a unique solution $u \in \mathcal{H}_p^1(\tau)$, and (2.5) holds.

Remark 2.10. Obviously (2.1) is stronger than (2.4) and (2.6) is stronger than (2.1). If $p = 2$ then Theorem 2.9 is true under the weaker condition (2.4) (see [14]), and this can be easily derived from Lemma 4.2(ii) and (3.8).

2.3. SPDEs with continuous coefficients on C^1 domains

Here we deal with an $L_q(L_p)$-theory of Eqs. (1.1) and (1.2) on bounded C^1 domains.

Assumption 2.11. The domain $\mathcal{O}$ is of class C_u^1. In other words, for any $x_0 \in \partial\mathcal{O}$, there exist constants $r, K \in (0, \infty)$ and a one-to-one continuously differentiable mapping Ψ of $B_r(x_0)$ onto a domain $J \subset \mathbb{R}^d$ such that

(i) $J_+ := \Psi(B_r(x_0) \cap \mathcal{O}) \subset \mathbb{R}_+^d$ and $\Psi(x_0) = 0$;
(ii) $\Psi(B_r(x_0) \cap \partial\mathcal{O}) = J \cap \{y \in \mathbb{R}^d : y^1 = 0\}$;
(iii) $\|\Psi\|_{C^1(B_r(x_0))} \leq K$ and $|\Psi^{-1}(y_1) - \Psi^{-1}(y_2)| \leq K|y_1 - y_2|$ for any $y_i \in J$;
(iv) Ψ_x is uniformly continuous in $B_r(x_0)$.

We use the Banach spaces introduced in [6,9] and [12]. Let ψ be an infinitely differentiable function defined in $\mathcal{O}$ such that

$$\rho(x) \leq N\psi(x) \leq N\rho(x)$$

and $\zeta \in C_0^\infty(\mathbb{R}_+)$ be a nonnegative function satisfying

$$\sum_{n=-\infty}^{\infty} \zeta(\mathrm{e}^{n+t}) > c > 0, \quad \forall t \in \mathbb{R}. \tag{2.7}$$

For $x \in \mathcal{O}$ and $n \in \mathbb{Z} = \{0, \pm 1, \dots\}$ define

$$\zeta_n(x) = \zeta(\mathrm{e}^n \psi(x)).$$

For $\gamma, \theta \in \mathbb{R}$, the weighted Sobolev space $H^\gamma_{p,\theta}(\mathcal{O})$ is defined as the set of all distributions u on $\mathcal{O}$ such that

$$\|u\|^p_{H^\gamma_{p,\theta}(\mathcal{O})} := \sum_{n=-\infty}^{\infty} \mathrm{e}^{n\theta} \|\zeta_{-n}(\mathrm{e}^n \cdot) u(\mathrm{e}^n \cdot)\|^p_{H^\gamma_p} < \infty. \tag{2.8}$$

If $g = (g^1, g^2, \dots)$ is an ℓ_2-valued function, we define

$$\|g\|^p_{H^\gamma_{p,\theta}(\mathcal{O})} := \sum_{n=-\infty}^{\infty} \mathrm{e}^{n\theta} \|\zeta_{-n}(\mathrm{e}^n \cdot) g(\mathrm{e}^n \cdot)\|^p_{H^\gamma_p(\ell_2)}.$$

We also introduce Banach space $H^\gamma_{p,\theta}$ defined on $\mathbb{R}^d_+$ by formally taking $\psi(x) = x^1$ so that $\zeta_n(x) = \zeta(\mathrm{e}^n x^1)$ and (2.8) becomes

$$\|u\|^p_{H^\gamma_{p,\theta}} := \sum_{n=-\infty}^{\infty} \mathrm{e}^{n\theta} \|\zeta(\cdot) u(\mathrm{e}^n \cdot)\|^p_{H^\gamma_p} < \infty.$$

It is known that the set $H^\gamma_{p,\theta}(\mathcal{O})$ is independent of the choice of ζ and ψ, and the norms generated by different choices of ζ and ψ are all equivalent. In particular, if γ is a nonnegative integer then

$$\|u\|^p_{H^\gamma_{p,\theta}(\mathcal{O})} \sim \sum_{|\alpha| \le \gamma} \int_{\mathcal{O}} |\rho^{|\alpha|} D^\alpha u|^p \rho^{\theta-d} \mathrm{d}x. \tag{2.9}$$

Also for any smooth function $\eta \in C_0^\infty(\mathbb{R}_+)$,

$$\sum_{n \in \mathbb{Z}} \mathrm{e}^{n\theta} \|\eta(\cdot) u(\mathrm{e}^n \cdot)\|^p_{H^\gamma_p} \le N \|u\|^p_{H^\gamma_{p,\theta}}. \tag{2.10}$$

Denote $\rho(x, y) = \rho(x) \wedge \rho(y)$. For $n \in \mathbb{Z}$, $\mu \in (0, 1]$ and $k = 0, 1, 2, \dots,$ define

$$[u]^{(n)}_k = \sup_{\substack{x \in \mathcal{O} \\ |\beta| = k}} \rho^{k+n}(x) |D^\beta u(x)|,$$

$$[u]^{(n)}_{k+\mu} = \sup_{\substack{x, y \in \mathcal{O} \\ |\beta| = k}} \rho^{k+\mu+n}(x, y) \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\mu},$$

$$|u|^{(n)}_k = \sum_{j=0}^{k} [u]^{(n)}_j, \quad |u|^{(n)}_{k+\mu} = |u|^{(n)}_k + [u]^{(n)}_{k+\mu}.$$

We collect some well known properties of the space $H^\gamma_{p,\theta}(\mathcal{O})$.

Lemma 2.12 (*[9,12]*). (i) *Assume that $\gamma - d/p = m + \nu$ for some $m = 0, 1, \dots$ and $\nu \in (0, 1]$. Let i, j be multi-indices such that $|i| \le m$, $|j| = m$. Then for any $u \in H^\gamma_{p,\theta}(\mathcal{O})$, we have*

$$\psi^{|i|+\theta/p} D^i u \in C(\mathcal{O}), \qquad \psi^{m+\nu+\theta/p} D^j u \in C^\nu(\mathcal{O}),$$
$$|\psi^{|i|+\theta/p} D^i u|_{C(\mathcal{O})} + [\psi^{m+\nu+\theta/p} D^j u]_{C^\nu(\mathcal{O})} \le N \|u\|_{H^\gamma_{p,\theta}(\mathcal{O})}.$$

(ii) *For any $\nu, \gamma \in \mathbb{R}$, $\psi^\nu H^\gamma_{p,\theta}(\mathcal{O}) = H^\gamma_{p,\theta-p\nu}(\mathcal{O})$, and*

$$\|u\|_{H^\gamma_{p,\theta-p\nu}(\mathcal{O})} \le N \|\psi^{-\nu} u\|_{H^\gamma_{p,\theta}(\mathcal{O})} \le N \|u\|_{H^\gamma_{p,\theta-p\nu}(\mathcal{O})}.$$

(iii) *$\psi D, D\psi : H^\gamma_{p,\theta}(\mathcal{O}) \to H^{\gamma-1}_{p,\theta}(\mathcal{O})$ are bounded linear operators, and for any $u \in \psi H^\gamma_{p,\theta}(\mathcal{O})$,*

$$\|\psi^{-1} u\|_{H^\gamma_{p,\theta}(\mathcal{O})} \le N \|u_x\|_{H^{\gamma-1}_{p,\theta}(\mathcal{O})} + N \|\psi^{-1} u\|_{H^{\gamma-1}_{p,\theta}(\mathcal{O})} \le N \|\psi^{-1} u\|_{H^\gamma_{p,\theta}(\mathcal{O})}.$$

Furthermore, if $\theta \ne d - 1 + p$, then

$$\|(x^1)^{-1} u\|_{H^\gamma_{p,\theta}} \le N \|u_x\|_{H^{\gamma-1}_{p,\theta}}.$$

(iv) *There exists a constant $N > 0$ such that*

$$\|au\|_{H^\gamma_{p,\theta}(\mathcal{O})} \le N |a|^{(0)}_{|\gamma|_+} \|u\|_{H^\gamma_{p,\theta}(\mathcal{O})}.$$

Denote

$$\mathbb{H}^{\gamma,q}_{p,\theta}(\mathcal{O}, \tau) = L_q((0, \tau]], \mathcal{P}, H^\gamma_{p,\theta}(\mathcal{O})), \qquad \mathbb{H}^{\gamma,q}_{p,\theta}(\tau) = L_q((0, \tau]], \mathcal{P}, H^\gamma_{p,\theta}),$$
$$\mathbb{L}^q_{p,\theta}(\mathcal{O}, \tau) = \mathbb{H}^{0,q}_{p,\theta}(\mathcal{O}, \tau), \qquad U^{\gamma,q}_{p,\theta}(\mathcal{O}) = \psi^{1-2/q} L_q(\mathcal{O}, \mathcal{F}_0, H^{\gamma-2/q}_{p,\theta}(\mathcal{O})),$$

where

$$\|u\|^q_{U^{\gamma,q}_{p,\theta}} := E \|\psi^{2/q-1} u\|^q_{H^{\gamma-2/q}_{p,\theta}(\mathcal{O})}.$$

By $\mathfrak{H}^{\gamma,q}_{p,\theta}(\mathcal{O}, \tau)$ we denote the space of all functions $u \in \psi \mathbb{H}^{\gamma,q}_{p,\theta}(\mathcal{O}, \tau)$ such that $u(0, \cdot) \in U^{\gamma,q}_{p,\theta}$ and for some $f \in \psi^{-1} \mathbb{H}^{\gamma-2,q}_{p,\theta}(\mathcal{O}, \tau)$, $g \in \mathbb{H}^{\gamma-1,q}_{p,\theta}(\mathcal{O}, \tau)$

$$\mathrm{d}u = f \mathrm{d}t + g^k \mathrm{d}w^k_t$$

in the sense of distributions. The norm in $\mathfrak{H}^{\gamma,q}_{p,\theta}(\mathcal{O}, \tau)$ is introduced by

$$\|u\|_{\mathfrak{H}^{\gamma,q}_{p,\theta}(\mathcal{O},\tau)} = \|\psi^{-1} u\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(\mathcal{O},\tau)} + \|\psi f\|_{\mathbb{H}^{\gamma-2,q}_{p,\theta}(\mathcal{O},\tau)} + \|g\|_{\mathbb{H}^{\gamma-2,q}_{p,\theta}(\mathcal{O},\tau)} + \|u(0)\|_{U^{\gamma,q}_{p,\theta}}.$$

In the above notation we drop q if $p = q$.

Assumption 2.13. For any ω, t, x,

$$\rho(x)(|\bar{b}^i(t, x)| + |b^i(t, x)| + \rho(x)|c(t, x)| + |\nu(t, x)|_{\ell_2}) \le K$$

and there is a control on the behavior of $\bar{b}, b, c$ and ν near $\partial\mathcal{O}$, namely,

$$\lim_{\substack{\rho(x)\to 0\\ x\in\mathcal{O}}} \sup_{\omega,t} \rho(x)(|\bar{b}^i(t, x)| + |b^i(t, x)| + \rho(x)|c(t, x)| + |\nu(t, x)|_{\ell_2}) = 0.$$

Remark 2.14. Assumption 2.13 allows the coefficients $\bar{b}^i$, b^i, c and ν to be unbounded and to blow up near the boundary of $\mathcal{O}$. For instance, it holds if for some $\varepsilon, N > 0$

$$|\bar{b}^i(t,x)| + |b^i(t,x)| + |\nu(t,x)|_{\ell_2} \le N\rho^{\varepsilon-1}(x), \quad |c(t,x)| \le N\rho^{\varepsilon-2}(x).$$

Theorem 2.15. *Let Assumptions 2.1, 2.11 and 2.13 be satisfied. Take $\kappa_0 \in (0,1)$ from Lemma 5.1 and assume*

$$d - \kappa_0 \le \theta < d - 1 + p. \tag{2.11}$$

Then, for any $f^i \in \mathbb{L}^q_{p,\theta}(\mathcal{O},\tau)$, $f \in \psi^{-1}\mathbb{H}^{-1,q}_{p,\theta}(\mathcal{O},\tau)$, $g \in \mathbb{L}^q_{p,\theta}(\mathcal{O},\tau)$ and $\psi^{2/q-1-\varepsilon_0}u_0 \in L_q(\Omega,\mathcal{F}_0,H^{1-2/q+\varepsilon_0}_{p,\theta}(\mathcal{O}))$, Eq. (1.2) with initial data u_0 has a unique solution $u \in \mathfrak{H}^{1,q}_{p,\theta}(\mathcal{O},\tau)$, and for this solution

$$\begin{aligned}\|u\|^q_{\mathfrak{H}^{1,q}_{p,\theta}(\mathcal{O},\tau)} &\le N\|f^i\|^q_{\mathbb{L}^q_{p,\theta}(\mathcal{O},\tau)} + N\|\psi f\|^q_{\mathbb{H}^{-1,q}_{p,\theta}(\mathcal{O},\tau)}\\ &\quad + N\|g\|^q_{\mathbb{L}^q_{p,\theta}(\mathcal{O},\tau)} + NE\|\psi^{2/q-1-\varepsilon_0}u_0\|^q_{H^{1-2/q+\varepsilon_0}_{p,\theta}(\mathcal{O})},\end{aligned} \tag{2.12}$$

where the constant N depends only on $d, p, q, \theta, \delta_0, K, T$ and $\mathcal{O}$.

Assumption 2.16. For each $\omega, t > 0$,

$$|a(t,\cdot)|^{(0)}_{|\gamma|_+} + |b(t,\cdot)|^{(1)}_{|\gamma|_+} + |c(t,\cdot)|^{(2)}_{|\gamma|_+} + |\sigma|^{(0)}_{|\gamma+1|_+} + |\nu|^{(1)}_{|\gamma+1|_+} \le K.$$

Theorem 2.17. *Let $\gamma \in \mathbb{R}$. Suppose that Assumptions 2.1, 2.11, 2.13 and 2.16 are satisfied. Also assume that (2.11) holds with κ_0 from Lemma 5.1. Then for any $f \in \psi^{-1}\mathbb{H}^{\gamma,q}_{p,\theta}(\mathcal{O},\tau)$, $g \in \mathbb{H}^{\gamma+1,q}_{p,\theta}(\mathcal{O},\tau)$ and $\psi^{2/q-1-\varepsilon_0}u_0 \in L_q(\Omega,\mathcal{F}_0,H^{\gamma+2-2/q+\varepsilon_0}_{p,\theta}(\mathcal{O}))$, Eq. (1.1) with initial data u_0 has a unique solution $u \in \mathfrak{H}^{\gamma+2,q}_{p,\theta}(\mathcal{O},\tau)$, and for this solution*

$$\begin{aligned}\|u\|^q_{\mathfrak{H}^{\gamma+2,q}_{p,\theta}(\mathcal{O},\tau)} &\le N\|\psi f\|^q_{\mathbb{H}^{\gamma,q}_{p,\theta}(\mathcal{O},\tau)} + N\|g\|^q_{\mathbb{H}^{\gamma+1,q}_{p,\theta}(\mathcal{O},\tau)}\\ &\quad + NE\|\psi^{2/q-1-\varepsilon_0}u_0\|^q_{H^{\gamma+2-2/q+\varepsilon_0}_{p,\theta}(\mathcal{O})},\end{aligned} \tag{2.13}$$

where the constant N depends only on $d, p, q, \theta, \delta, K, T$ and $\mathcal{O}$.

The following results illustrate advantages of $L_q(L_p)$-theory over L_p-theory. (2.14) follows from Lemma 2.12(i), and (2.15) is a consequence of (4.10) and Theorem 4.11 in [6].

Corollary 2.18. (i) *Let $1 - d/p =: \mu > 0$ and $u \in \mathfrak{H}^1_{p,\theta}(\mathcal{O},\tau)$, then*

$$E\int_0^\tau (|\psi^{\theta/p-1}u|^p_{C^0(\mathcal{O})} + [\psi^{(\theta-d)/p}u]^p_{C^\mu(\mathcal{O})})\mathrm{d}t < \infty. \tag{2.14}$$

(ii) *If $u \in \mathfrak{H}^{\gamma+2,q}_{p,\theta}(\mathcal{O},\tau)$ and*

$$2/q < \alpha < \beta \le 1, \quad \gamma + 2 - \beta - d/p = k + \varepsilon$$

where $k = 0, 1, \ldots, \varepsilon \in (0, 1]$, *then for* $\nu := \beta - 1 + \theta/p$ *and multi-indices* i *and* j *such that* $|i| \le k$ *and* $|j| = k$, *we have*

$$E \sup_{t \neq s} |t-s|^{-(\alpha q/2-1)} \big(|\psi^{\nu+|i|} D^i (u(t)-u(s))|^q_{C(\mathcal{O})} \\ + [\psi^{\nu+|j|+\varepsilon} D^j (u(t)-u(s))]^q_{C^\varepsilon(\mathcal{O})} \big) < \infty. \tag{2.15}$$

In particular, if $u_0 = 0, \gamma \ge -1, \theta \le d$ *and* $r_0 := 1 - 2/q - d/p > 0$, *then for any* $r \in (0, r_0)$

$$E \sup_{t \le \tau} \sup_{x,y \in \mathcal{O}} \frac{|u(t,x)-u(t,y)|^q}{|x-y|^{rq}} < \infty \tag{2.16}$$

$$E \sup_{x \in \mathcal{O}} \sup_{t,s \le \tau} \frac{|u(t,x)-u(s,x)|^q}{|t-s|^{rq/2}} < \infty. \tag{2.17}$$

2.4. SPDEs with measurable coefficients on C^1 domains

In this section we give an L_p-theory of Eq. (1.2) with measurable coefficients on bounded C^1 domains.

Theorem 2.19. *Let Assumptions* 2.11 *and* 2.13 *and* (2.4) *hold. Also assume*

$$d - \frac{1}{2K/\delta_0 - 1} < \theta < d + \frac{1}{2K/\delta_0 + 1}. \tag{2.18}$$

Then for any $f^i, g \in \mathbb{L}_2(\mathcal{O}, \tau)$, $f \in \psi^{-1}\mathbb{H}^{-1}_{2,\theta}(\mathcal{O}, \tau)$ *and* $u_0 \in U^1_{2,\theta}(\mathcal{O})$ *Eq.* (1.2) *has a unique solution* $u \in \mathfrak{H}^1_{2,\theta}(\mathcal{O}, \tau)$, *and*

$$\|u\|_{\mathfrak{H}^1_{2,\theta}(\mathcal{O},\tau)} \le N(\|f^i\|_{\mathbb{L}_{2,\theta}(\mathcal{O},\tau)} + \|\psi f\|_{\mathbb{H}^{-1}_{2,\theta}(\mathcal{O},\tau)} + \|g\|_{\mathbb{L}_{2,\theta}(\mathcal{O},\tau)} + \|u_0\|_{U^1_{2,\theta}(\mathcal{O})}),$$

where $N = N(\delta_0, K, \theta, \mathcal{O}, T)$.

Theorem 2.20. *Let the assumptions in Theorem* 2.19 *hold and* σ^i *be uniformly continuous in* x. *Then there exist constants* $p_0 > 2$, $\kappa_1 \in (0,1)$ *(depending only on* d, δ_0, K*) such that if* $p \in [2, p_0)$ *and*

$$(d-\kappa_1) \wedge \left(d + p - 2 - \frac{1}{\frac{Kp}{\delta_0(p-1)} - 1} \right) < \theta < d + p - 2 + \frac{1}{\frac{Kp}{\delta_0(p-1)} + 1} \tag{2.19}$$

then (a) *for any* $f^i, g \in \mathbb{L}_p(\mathcal{O}, \tau)$, $f \in \psi^{-1}\mathbb{H}^{-1}_{p,\theta}(\mathcal{O}, \tau)$ *and* $u_0 \in U^1_{p,\theta}(\mathcal{O})$ *Eq.* (1.2) *admits a unique solution* $u \in \mathfrak{H}^1_{p,\theta}(\mathcal{O}, \tau)$; (b) *for this solution*

$$\|u\|_{\mathfrak{H}^1_{p,\theta}(\mathcal{O},\tau)} \le N(\|f^i\|_{\mathbb{L}_{p,\theta}(\mathcal{O},\tau)} + \|\psi f\|_{\mathbb{H}^{-1}_{p,\theta}(\mathcal{O},\tau)} + \|g\|_{\mathbb{L}_{p,\theta}(\mathcal{O},\tau)} + \|u_0\|_{U^1_{p,\theta}(\mathcal{O})}), \tag{2.20}$$

where $N = N(\delta_0, K, \theta, \mathcal{O}, T)$.

The continuity of σ is dropped in the following result.

Theorem 2.21. *Suppose that Assumptions* 2.11 *and* 2.13 *and* (2.6) *are satisfied. Then there exists* $p_1 > 2, \kappa_1 \in (0,1)$ *(depending only on* d, δ_i, K*) such that assertions* (a), (b) *in Theorem* 2.20 *hold true if* $p \in [2, p_1)$ *and* (2.19) *holds.*

Remark 2.22. See Corollary 2.18 for Hölder estimates of the solutions of Eq. (1.2).

3. Proof of Theorems 2.3 and 2.5

First we prove the following lemmas.

Lemma 3.1. *For $i = 1, 2, \dots, n$, let $a^{(i)} = (a^{i,rs})$ and $\sigma^{(i)} = (\sigma^{i,rk})$ be independent of x and satisfy (2.1). Let $u^{(i)} \in \mathcal{H}^{\gamma+2}_{p,0}(T)$ be a solution of the equation*

$$\mathrm{d}u^{(i)} = (a^{i,rs} u^{(i)}_{x^r x^s} + f^{(i)})\mathrm{d}t + (\sigma^{i,rk} u^{(i)}_{x^r} + g^{(i)k})\mathrm{d}w^k_t. \tag{3.1}$$

Then

$$\begin{aligned} E\int_0^T \prod_{i=1}^n \|u^{(i)}_{xx}(t)\|^p_{H^\gamma_p}\mathrm{d}t &\le N\sum_{i=1}^n E\int_0^T \|f^{(i)}(t)\|^p_{H^\gamma_p}\prod_{j\neq i}\|u^{(j)}_{xx}(t)\|^p_{H^\gamma_p}\mathrm{d}t \\ &\quad + N\sum_{i=1}^n E\int_0^T \|g^{(i)}(t)\|^p_{H^{\gamma+1}_p}\prod_{j\neq i}\|u^{(j)}_{xx}(t)\|^p_{H^\gamma_p}\mathrm{d}t \\ &\quad + N\sum_{1\le i<j\le n} E\int_0^T \|g^{(i)}(t)\|^p_{H^{\gamma+1}_p}\|g^{(j)}(t)\|^p_{H^{\gamma+1}_p}\prod_{\ell\neq i,j}\|u^{(\ell)}_{xx}(t)\|^p_{H^\gamma_p}\mathrm{d}t. \end{aligned} \tag{3.2}$$

where $N = N(d, p, n, \delta)$.

Proof. See Lemma 2.3 in [7]. Actually in [7] this lemma is proved when $a := a^{(1)} = a^{(i)}$ and $\sigma := \sigma^{(1)} = \sigma^{(i)}$ $(i = 2, 3, \dots, n)$. But one can easily check that the proof there still holds in our case. □

Lemma 3.2. *For $i = 1, 2, \dots, n$, let $a^{(i)} = (a^{i,rs})$ and $\sigma^{(i)} = (\sigma^{i,rk})$ satisfy (2.1). Let Assumption 2.4 hold and*

$$|a^{i,rs}(t,x) - a^{i,rs}(t,y)| + |\sigma^{i,r}(t,x) - \sigma^{i,r}(t,y)|_{\ell_2} \le \beta, \quad \forall i, \omega, t, x, y.$$

Then there exists a constant $\beta_0 \in (0,\infty)$ depending only on d, p, q, δ, γ (independent of C) such that if $\beta \le \beta_0$ and $u^{(i)} \in \mathcal{H}^{\gamma+2}_{p,0}(T)$ is a solution of Eq. (3.1), then

$$\begin{aligned} E\int_0^T \prod_{i=1}^n \|u^{(i)}(t)\|^p_{H^{\gamma+2}_p}\mathrm{d}t &\le N\sum_{i=1}^n E\int_0^T \|f^{(i)}(t)\|^p_{H^\gamma_p}\prod_{j\neq i}\|u^{(j)}(t)\|^p_{H^{\gamma+2}_p}\mathrm{d}t \\ &\quad + N\sum_{i=1}^n E\int_0^T \|g^{(i)}(t)\|^p_{H^{\gamma+1}_p}\prod_{j\neq i}\|u^{(j)}(t)\|^p_{H^{\gamma+2}_p}\mathrm{d}t \\ &\quad + N\sum_{1\le i<j\le n} E\int_0^T \|g^{(i)}(t)\|^p_{H^{\gamma+1}_p}\|g^{(j)}(t)\|^p_{H^{\gamma+1}_p}\prod_{\ell\neq i,j}\|u^{(\ell)}(t)\|^p_{H^{\gamma+2}_p}\mathrm{d}t \\ &\quad + N\sum_{J\in\Gamma} E\int_0^T \left(\prod_{i\in J}\|u^{(i)}(t)\|^p_{H^{\gamma+2}_p}\right)\left(\prod_{j\notin J}\|u^{(j)}(t)\|^p_{H^\gamma_p}\right)\mathrm{d}t, \end{aligned}$$

where Γ is the collection of all subsets A of $\{1, 2, \dots, n\}$ such that $A \neq \{1, 2, \dots, n\}$ and $N = N(d, p, n, \gamma, \delta, C)$.

Proof. Denote

$$a_0^{(i)}(t,x)=a^{(i)}(t,0),\qquad \sigma_0^{(i)}(t,x)=\sigma^{(i)}(t,0),$$
$$f_0^{(i)}=(a^{i,rs}-a_0^{i,rs})u^{(i)}_{x^rx^s}+f^{(i)},\qquad g_0^{(i)k}=(\sigma^{i,rk}-\sigma_0^{i,rk})u_{x^r}+g^{(i)k},$$
$$C_0=\sup_{i,r,s,\omega,t}(|a^{i,rs}-a_0^{i,rs}|_{B^{|\gamma|_+}}+|\sigma^{i,r}-\sigma_0^{i,r}|_{B^{|\gamma+1|_+}}).$$

Then by Lemma 3.1, (3.2) holds with f_0^k and g_0^{ik} instead of f^i and g^{ik}, respectively. By Lemma 5.2 in [8]

$$\|(a^{i,rs}-a_0^{i,rs})u^{(i)}_{x^rx^s}\|_{H^\gamma_p}+\|(\sigma^{i,r}-\sigma_0^{i,r})u^{(i)}_{x^r}\|_{H^{\gamma+1}_p}\le NC_0\|u^{(i)}\|_{H^{\gamma+2}_p}.$$

Thus

$$\begin{aligned}
E\int_0^T\prod_{i=1}^n\|u^{(i)}(t)\|^p_{H^{\gamma+2}_p}\mathrm{d}t&\le NE\int_0^T\prod_{i=1}^n(\|u^{(i)}_{xx}(t)\|^p_{H^\gamma_p}+\|u^{(i)}(t)\|^p_{H^\gamma_p})\,\mathrm{d}t\\
&\le NE\int_0^T\left[\prod_{i=1}^n\|u^{(i)}_{xx}(t)\|^p_{H^{\gamma+2}_p}+\sum_{J\in\Gamma}\left(\prod_{i\in J}\|u^{(i)}_{xx}(t)\|^p_{H^\gamma_p}\right)\left(\prod_{j\notin J}\|u^{(j)}(t)\|^p_{H^\gamma_p}\right)\right]\mathrm{d}t\\
&\le N(C_0\vee C_0^2)E\int_0^T\prod_{i=1}^n\|u^{(i)}(t)\|^p_{H^{\gamma+2}_p}\mathrm{d}t\\
&\quad+N\sum_{i=1}^nE\int_0^T\|f^{(i)}(t)\|^p_{H^\gamma_p}\prod_{j\ne i}\|u^{(j)}(t)\|^p_{H^{\gamma+2}_p}\mathrm{d}t\\
&\quad+N\sum_{i=1}^nE\int_0^T\|g^{(i)}(t)\|^p_{H^{\gamma+1}_p}\prod_{j\ne i}\|u^{(j)}(t)\|^p_{H^{\gamma+2}_p}\mathrm{d}t\\
&\quad+N\sum_{1\le i<j\le n}E\int_0^T\|g^{(i)}(t)\|^p_{H^{\gamma+1}_p}\|g^{(j)}(t)\|^p_{H^{\gamma+1}_p}\prod_{\ell\ne i,j}\|u^{(\ell)}(t)\|^p_{H^{\gamma+2}_p}\mathrm{d}t\\
&\quad+N\sum_{J\in\Gamma}E\int_0^T\left(\prod_{i\in J}\|u^{(k)}(t)\|^p_{H^{\gamma+2}_p}\right)\left(\prod_{j\notin J}\|u^{(\ell)}(t)\|^p_{H^\gamma_p}\right)\mathrm{d}t.
\end{aligned}$$

Thus our lemma holds true if $N(C_0\vee C_0^2)<1/2$. Denote $a_m^{(i)}(t,x):=a^{(i)}(t/m^2,x/m)$ and $\sigma_m^{(i)}(t,x):=\sigma^{(i)}(t/m^2,x/m)$. Then we have

$$|a_m^{(i)}(t,\cdot)-a_m^{(i)}(t,0)|_{B^{|\gamma|_+}}\le\beta+m^{-(|\gamma|_+\wedge1)}C_0,$$

and we can drop the second term on the right if $\gamma=0$. Also we have a similar inequality for $\sigma_m^{(i)}$. Observe that $u_m^{(i)}(t,x):=u^{(i)}(t/m^2,x/m)$ satisfies

$$\mathrm{d}u_m^{(i)}=(a_m^{i,rs}u^{(i)}_{mx^rx^s}+f_m^{(i)})\mathrm{d}t+(\sigma_m^{i,rk}u^{(i)}_{mx^r}+g_m^{(i)k})\mathrm{d}w_t^k(m),$$

where $w_t^k(m):=mw^k_{t/m^2}$, $k=1,2,\dots$, are independent one-dimensional Wiener processes, $f_m^{(i)}(t,x):=m^{-2}f^{(i)}(t/m^2,x/m)$ and $g_m^{(i)k}(t,x):=m^{-1}g^{(i)k}(t/m^2,x/m)$. Then it follows that for β sufficiently small and m sufficiently large, the statements of the lemma hold if we replace $a^{(i)}$, $\sigma^{(i)}$, $u^{(i)}$, $f^{(i)}$, $g^{(i)}$, w_t^k and T by $a_m^{(i)}$, $\sigma_m^{(i)}$, $u_m^{(i)}$, $f_m^{(i)}$, $g_m^{(i)}$, $w_t^k(m)$ and m^2T, respectively.

Finally it suffices to observe that $\|\cdot\|_{H^\gamma_p}$ norms of $u^{(i)}(t/m^2, x/m)$ and $u^{(i)}(t,x)$ are comparable. The lemma is proved. $\square$

Lemma 3.3. *For $i = 1, 2, \dots, n$, let $a^{(i)} = (a^{i,rs})$ and $\sigma^{(i)} = (\sigma^{i,rk})$ satisfy (2.1) and*

$$|a^{i,rs}(t,x) - a^{i,rs}(t,y)| + |\sigma^{i,r}(t,x) - \sigma^{i,r}(t,y)|_{\ell_2} \le \beta, \quad \forall i, \omega, t, x, y.$$

Let $\bar f^{(i)} = (\bar f^{(i)1}, \dots, \bar f^{(i)d}) \in (\mathbb{L}_p(T))^d$, $f^{(i)} \in \mathbb{H}^{-1}_p(T)$ and $u^{(i)} \in \mathcal{H}^1_{p,0}(T)$ be a solution of the equation

$$\mathrm{d}u^{(i)}_t = (D_r(a^{i,rs}u^{(i)}_{x^s} + \bar f^{(i)r}) + f^{(i)})\mathrm{d}t + (\sigma^{i,rk}u^{(i)}_{x^r} + g^{(i)k})\mathrm{d}w^k_t.$$

Then there exists a constant $\beta_1 \in (0,1)$ depending only on d, p, q, δ such that if $\beta \le \beta_1$, then

$$\begin{aligned} E\int_0^T \prod_{i=1}^n \|u^{(i)}(t)\|^p_{H^1_p}\mathrm{d}t \le{}& N\sum_{i=1}^n E\int_0^T \|\bar f^{(i)}(t)\|^p_{L_p}\prod_{j\ne i}\|u^{(j)}(t)\|^p_{H^1_p}\mathrm{d}t \\ &+ \sum_{i=1}^n E\int_0^T (\|f^{(i)}(t)\|^p_{H^{-1}_p} + \|g^{(i)}(t)\|^p_{L_p})\prod_{j\ne i}\|u^{(j)}(t)\|^p_{H^1_p}\mathrm{d}t \\ &+ N\sum_{1\le i<j\le n} E\int_0^T \|g^{(i)}(t)\|^p_{L_p}\|g^{(j)}(t)\|^p_{L_p}\prod_{\ell\ne i,j}\|u^{(\ell)}(t)\|^p_{H^1_p}\mathrm{d}t \\ &+ N\sum_{J\in\Gamma} E\int_0^T \left(\prod_{i\in J}\|u^{(i)}(t)\|^p_{H^1_p}\right)\left(\prod_{j\notin J}\|u^{(j)}(t)\|^p_{L_p}\right)\mathrm{d}t, \end{aligned}$$

where $N = N(d, p, n, \delta)$.

Proof. Denote $a^{(i)}_0(t,x) = a^{(i)}(t,0)$, $\sigma^{(i)}_0(t,x) = \sigma^{(i)}(t,0)$ and

$$f^{(i)}_0 = D_r((a^{i,rs} - a^{i,rs}_0)u^{(i)}_{x^s} + \bar f^{(i)r}) + f^{(i)},$$
$$g^{(i)k}_0 = (\sigma^{i,rk} - \sigma^{i,rk}_0)u^{(i)}_{x^r} + g^{(i)k}.$$

Then by Lemma 3.1, (3.2) holds with $\gamma = -1$, $f^{(i)}_0$ and $g^{(i)}_0$ (instead of $f^{(i)}$ and $g^{(i)}$, respectively). As easy to check

$$\|f^{(i)}_0\|_{H^{-1}_p} \le N\beta\|u^{(i)}\|_{H^1_p} + N\|\bar f^{(i)}\|_{L_p} + \|f^{(i)}\|_{H^{-1}_p}, \tag{3.3}$$

$$\|g^{(i)}_0\|_{L_p} \le \beta\|u^{(i)}_x\|_{L_p} + \|g^{(i)}\|_{L_p} \le N\beta\|u^{(i)}\|_{H^1_p} + \|g^{(i)}\|_{L_p}. \tag{3.4}$$

Now it suffices to repeat the arguments in the proof of Lemma 3.2 using (3.3) and (3.4). $\square$

Proof of Theorem 2.5. The theorem is already known (see [7]) if $a = I$ and $b^i = c = \sigma^{ik} = \nu^k = 0$. Thus considering the method of continuity (see, for instance, the proof of Theorem 5.1 in [8]), we convince ourselves that to prove the theorem it suffices to show that a priori estimate (2.3) holds given that a solution $u \in \mathcal{H}^{\gamma+2,q}_p(\tau)$ already exists. As usual we assume that $\tau \equiv T$ and $u_0 = 0$. Also we assume that $q = np$ for some positive integer n. The case $q \ne np$ is covered by the Marcinkiewicz interpolation theorem.

Choose $r > 0$ such that

$$|a^{ij}(t,x) - a^{ij}(t,y)| + |\sigma^i(t,x) - \sigma^i(t,y)|_{\ell_2} < \beta_0/2$$

whenever $|x-y| < r$. Let $\{\zeta_m : m = 1, 2, \ldots\}$ be a standard partition of unity such that for any m the support of ζ_m lies in a ball $B_{r/4}(x_m)$. Also for each m, choose a function $\eta_m \in C_0^\infty(B_{r/2}(x_m))$ such that $0 \le \eta_m \le 1$ and $\eta_m = 1$ on the support of ζ_m. Denote

$$a^{(m)}(t,x) = (a^{m,ij}(t,x)) := a(t,x)\eta_m(x) + (1-\eta_m(x))a(t,x_m),$$
$$\sigma^{(m)}(t,x) = (\sigma^{m,ik}(t,x)) = \sigma(t,x)\eta_m(x) + (1-\eta_m(x))\sigma(t,x_m).$$

Then one can easily check that for each m, (a_m, σ_m) satisfies (2.1) and

$$\sup_{\omega,t,x,y} (|a^{(m)}(t,x) - a^{(m)}(t,y)| + |\sigma^{(m)}(t,x) - \sigma^{(m)}(t,y)|_{\ell_2}) < \beta_0.$$

By Lemma 6.7 in [8],

$$\begin{aligned} E\int_0^T \|u\|^{np}_{H_p^{\gamma+2}} \mathrm{d}t &\le NE\int_0^T \left(\sum_m \|\zeta_m u\|^p_{H_p^{\gamma+2}}\right)^n \mathrm{d}t \\ &= N\sum_{m_1,m_2,\ldots,m_n} E\int_0^T \prod_{i=1}^n \|\zeta_{m_i} u\|^p_{H_p^{\gamma+2}} \mathrm{d}t. \end{aligned} \tag{3.5}$$

Note that $\zeta_{m_i} u$ satisfies

$$d(\zeta_{m_i} u) = (a^{m_i,rs}(\zeta_{m_i} u)_{x^r x^s} + f^{(m_i)})\mathrm{d}t + (\sigma^{m_i,rk}(\zeta_{m_i} u)_{x^r} + g^{(m_i)k})\mathrm{d}w_t^k$$

where

$$f^{(m_i)} := -2a^{rs} u_{x^r} \zeta_{m_i x^r} - a^{rs} u \zeta_{m_i x^r x^s} + b^r u_{x^r} \zeta_{m_i} + cu\zeta_{m_i} + \zeta_{m_i} f,$$
$$g^{(m_i)k} = \sigma^{m_i,rk} u \zeta_{m_i x^r} + \nu^k u \zeta_{m_i} + g^k \zeta_{m_i}.$$

By Lemma 5.2 in [8]

$$\begin{aligned} \|f^{(m_i)}\|_{H_p^\gamma} \le\ & N\|u_{x^r}\zeta_{m_i x^s}\|_{H_p^\gamma} + N\|u\zeta_{m_i x^r x^s}\|_{H_p^\gamma} + N\|u_{x^r}\zeta_{m_i}\|_{H_p^\gamma} \\ & + N\|u\zeta_{m_i}\|_{H_p^\gamma} + \|\zeta_{m_i} f\|_{H_p^\gamma}, \end{aligned} \tag{3.6}$$

$$\|g^{(m_i)}\|_{H_p^{\gamma+1}} \le N\|u\zeta_{m_i x^r}\|_{H_p^{\gamma+1}} + N\|u\zeta_{m_i}\|_{H_p^{\gamma+1}} + \|g\zeta_{m_i}\|_{H_p^{\gamma+1}}. \tag{3.7}$$

By Lemma 6.7 in [8] and Lemma 3.2, for any $t \le T$,

$$\begin{aligned} &\sum_{m_1,m_2,\ldots,m_n} E\int_0^t \prod_{i=1}^n \|\zeta_{m_i} u\|^p_{H_p^{\gamma+2}} \mathrm{d}s \\ &\le N\sum_{i=1}^n \sum_{m_1,m_2,\ldots,m_n} E\int_0^t \|f^{(m_i)}\|^p_{H_p^\gamma} \prod_{j\ne i} \|u\zeta_{m_j}\|^p_{H_p^{\gamma+2}} \mathrm{d}s \\ &\quad + N\sum_{i=1}^n \sum_{m_1,m_2,\ldots,m_n} E\int_0^t \|g^{(m_i)}\|^p_{H_p^{\gamma+1}} \prod_{j\ne i} \|u\zeta_{m_j}\|^p_{H_p^{\gamma+2}} \mathrm{d}s \\ &\quad + N\sum_{1\le i<j\le n} \sum_{m_1,m_2,\ldots,m_n} E\int_0^t \|g^{(m_i)}\|^p_{H_p^{\gamma+1}} \|g^{(m_j)}\|^p_{H_p^{\gamma+1}} \prod_{\ell\ne i,j} \|u\zeta_{m_\ell}\|^p_{H_p^{\gamma+2}} \mathrm{d}s \\ &\quad + N\sum_{J\in\Gamma} \sum_{m_1,m_2,\ldots,m_n} E\int_0^t \left(\prod_{i\in J} \|u\zeta_{m_i}\|^p_{H_p^{\gamma+2}}\right)\left(\prod_{j\notin J} \|u\zeta_{m_j}\|^p_{H_p^\gamma}\right) \mathrm{d}s. \end{aligned}$$

Here,

$$\sum_{i=1}^{n}\sum_{m_1,m_2,\ldots,m_n} E\int_0^t \|u_{x^r}\zeta_{m_i x^s}\|^p_{H^\gamma_p}\prod_{j\neq i}\|u\zeta_{m_j}\|^p_{H^{\gamma+2}_p}\mathrm{d}s$$

$$=\sum_{i=1}^{n}E\int_0^t\left(\sum_{m_i}\|u_{x^r}\zeta_{m_i x^s}\|^p_{H^\gamma_p}\right)\prod_{j\neq i}\left(\sum_{m_j}\|u\zeta_{m_j}\|^p_{H^{\gamma+2}_p}\right)\mathrm{d}s$$

$$\leq NE\int_0^t\|u_x\|^p_{H^\gamma_p}\|u\|^{(n-1)p}_{H^{\gamma+2}_p}\mathrm{d}s\leq\varepsilon\|u\|^{np}_{\mathbb{H}^{\gamma+2,np}_p(t)}+N\|u\|^{np}_{\mathbb{H}^{\gamma+1,np}_p(t)}.$$

Using similar computation and coming back to (3.5) we get

$$\|u\|^{np}_{\mathbb{H}^{\gamma+2,np}_p(t)}\leq N\|u\|^{np}_{\mathbb{H}^{\gamma+1,np}_p(t)}+N\|f\|^{np}_{\mathbb{H}^{\gamma,np}_p(t)}+N\|g\|^{np}_{\mathbb{H}^{\gamma+1,np}_p(t)}.$$

Thus for each $t\leq T$,

$$\|u\|^{np}_{\mathcal{H}^{\gamma+2,np}_p(t)}\leq N\|u\|^{np}_{\mathbb{H}^{\gamma+1,np}_p(t)}+N\|f\|^{np}_{\mathbb{H}^{\gamma,np}_p(T)}+N\|g\|^{np}_{\mathbb{H}^{\gamma+1,np}_p(T)}.$$

Now we use

$$\|u\|^{np}_{\mathbb{H}^{\gamma+1,np}_p(t)}\leq N(d,p,q,\gamma,T)\int_0^t\|u\|^{np}_{\mathcal{H}^{\gamma+2,np}_p(s)}\mathrm{d}s. \tag{3.8}$$

This inequality follows from Theorems 7.1 and 7.2 in [8] if $p=q$, and from Corollary 4.12 in [6] if $q>p$. Consequently, for each $t\leq T$,

$$\|u\|^{np}_{\mathcal{H}^{\gamma+2,np}_p(t)}\leq N\int_0^t\|u\|^{np}_{\mathcal{H}^{\gamma+2,np}_p(s)}\mathrm{d}s+N\|f\|^{nq}_{\mathbb{H}^{\gamma,np}_p(T)}+N\|g\|^{np}_{\mathbb{H}^{\gamma+1,np}_p(T)}.$$

Finally, Gronwall's inequality leads to (2.3). The theorem is proved.

Proof of Theorem 2.3. Again we only establish a priori (2.2) given that $\tau\equiv T, u_0=0$ and $q=np$. Also we assume that $b^i\equiv 0$. For the case $b^i\not\equiv 0$, we only refer to the proof of Theorem 2.12 in [2].

Define a partition of unity $\{\zeta_m : m=1,2\ldots\}$, $a^{(m)}$ and $\sigma^{(m)}$ as in the proof of Theorem 2.5 such that

$$\sup_{\omega,t,x,y}(|a^{(m)}(t,x)-a^{(m)}(t,y)|+|\sigma^{(m)}(t,x)-\sigma^{(m)}(t,y)|_{\ell_2})<\beta_1.$$

As in (3.5),

$$E\int_0^t\|u\|^{np}_{H^1_p}\mathrm{d}s\leq N\sum_{m_1,\ldots,m_n}E\int_0^t\prod_{i=1}^{n}\|\zeta_{m_i}u\|^p_{H^1_p}\mathrm{d}s.$$

Note that $u^{(m_i)}:=\zeta_{m_i}u$ satisfies

$$\mathrm{d}u^{(m_i)}=(D_r(a^{m_i,rs}u^{(m_i)}_{x^s}+\bar f^{(m_i)r})+f^{(m_i)})\mathrm{d}t+(\sigma^{m_i,rk}u^{(m_i)}_{x^r}+g^{(m_i)k})\mathrm{d}w^k_t,$$

where

$$\bar f^{(m_i)r}:=\bar b^r\zeta_{m_i}u+f^r\zeta_{m_i}-a^{rs}u\zeta_{m_i x^s},$$
$$f^{(m_i)}:=-a^{rs}u_{x^s}\zeta_{m_i x^r}-\bar b^r u\zeta_{m_i x^r}-f^r\zeta_{m_i x^r}+c\zeta_{m_i}u+f\zeta_{m_i},$$
$$g^{(m_i)k}=-\sigma^{rk}u\zeta_{m_i x^r}+\nu^k u\zeta_{m_i}+g^k\zeta_{m_i}.$$

Obviously,

$$\|\bar{f}^{(m_i)r}\|_{L_p} \leq N\|\zeta_{m_i}u\|_{L_p} + \|f^r\zeta_{m_i}\|_{L_p} + \|u\zeta_{m_i x^s}\|_{L_p},$$
$$\|f^{(m_i)}\|_{H_p^{-1}} \leq \|a^{rs}u_{x^s}\zeta_{m_i x^r}\|_{H_p^{-1}} + N\|u\zeta_{m_i x^r}\|_{L_p} + \|f^r\zeta_{m_i x^r}\|_{L_p} + N\|\zeta_{m_i}u\|_{L_p} + \|f\zeta_{m_i}\|_{H_p^{-1}},$$

and

$$\|g^{(m_i)}\|_{L_p} \leq N\|u\zeta_{m_i x^r}\|_{L_p} + N\|u\zeta_{m_i}\|_{L_p} + \|g\|_{L_p}.$$

Here we claim that for any $\varepsilon > 0$,

$$\|a^{rs}u_{x^s}\zeta_{m_i x^r}\|_{H_p^{-1}} \leq \varepsilon\|u_{x^s}\zeta_{m_i x^r}\|_{L_p} + N(\varepsilon)\|u_{x^s}\zeta_{m_i x^r}\|_{H_p^{-1}}. \tag{3.9}$$

Indeed, (remember that a is uniformly continuous) take a sequence of smooth functions a_n satisfying

$$\sup_{\omega,t,x}|a - a_n| \leq 1/n.$$

Then

$$\|a^{rs}u_{x^s}\zeta_{m_i x^r}\|_{H_p^{-1}} \leq \|a_n^{rs}u_{x^s}\zeta_{m_i x^r}\|_{H_p^{-1}} + \|(a^{rs} - a_n^{rs})u_{x^s}\zeta_{m_i x^r}\|_{L_p}$$
$$\leq N|a_n|_{C^1}\|u_{x^s}\zeta_{m_i x^r}\|_{H_p^{-1}} + 1/n\|u_{x^s}\zeta_{m_i x^r}\|_{L_p}.$$

Now we use Lemma 3.3 to estimate

$$\sum_{m_1,\ldots,m_n} E\int_0^T \prod_{i=1}^n \|\zeta_{m_i}u\|_{H_p^1}^p \mathrm{d}t.$$

Similar computations as in the proof of Theorem 2.8 show that

$$\|u\|^{np}_{\mathbb{H}_p^{1,np}(t)} \leq N\varepsilon\|u\|^{np}_{\mathbb{H}_p^{1,np}(t)} + N(\varepsilon)\|u\|^{np}_{\mathbb{L}_p^{np}(t)} + N\|f^i\|^{np}_{\mathbb{L}_p^{np}(T)} + N\|f\|^{np}_{\mathbb{H}_p^{-1,np}(T)} + \|g\|^{np}_{\mathbb{L}_p^{np}(T)}.$$

This, (3.8) and Gronwall's inequality certainly yield (2.2).

4. Proof of Theorems 2.8 and 2.9

Consider the equation

$$\mathrm{d}u = [D_i(a^{ij}u_{x^j} + f^i) + f]\mathrm{d}t + (\sigma^{ik}u_{x^i} + g^k)\mathrm{d}w_t^k. \tag{4.1}$$

Lemma 4.1. (i) *Let (2.4) hold with $\sigma^{ik} = 0$. In other words,*

$$\delta_0 I < (a^{ij}) < KI. \tag{4.2}$$

There exists $p_0 = p_0(d, \delta_0, K) > 2$ such that if $p \in [2, p_0)$ then Eq. (4.1) with $f = \sigma^{ik} = 0$ has a unique solution $u \in \mathcal{H}^1_{p,0}(T)$, and for this solution

$$\|u_x\|_{\mathbb{L}_p(T)} \leq N(p)(\|f^i\|_{\mathbb{L}_p(T)} + \|g\|_{\mathbb{L}_p(T)}), \tag{4.3}$$

where $N(p) = N(d, p, \delta_0, K)$.

(ii) *Let* (2.4) *hold and* σ^i *be independent of* x. *Then* (4.3) *holds if* $p \in [2, p_0)$ *and* $u \in \mathcal{H}^1_{p,0}(T)$ *is a solution of Eq.* (4.1) *with* $f = 0$.

(iii) *Let* (2.4) *hold, then* (4.3) *holds if* $p = 2$ *and* $u \in \mathcal{H}^1_{2,0}(T)$ *is a solution of Eq.* (4.1) *with* $f = 0$.

Proof. The first assertion is due to Yoo ([15], Theorem 3.2). We only mention that using a scaling argument (consider $u(Tt, \sqrt{T}x)$) one can easily check that the constant $N(p)$ is independent of T.

The second assertion is a consequence of Lemma 4.7 in [8]. Indeed, define

$$\xi_t = \int_0^t \sigma^{ik} \mathrm{d}w^k_t \in \mathbb{R}^d, \qquad \bar{a}(t,x) = a(t, x - \xi_t), \qquad \bar{u}(t,x) = u(t, x - \xi_t)$$

and define $\bar{f}^i$, $\bar{g}^k$ similarly. Then $\bar{u}$ satisfies

$$\mathrm{d}\bar{u} = D_i((\bar{a}^{ij} - \alpha^{ij})\bar{u}_{x^j} + \bar{f}^i - \sigma^{ik}\bar{g}^k)\mathrm{d}t + \bar{g}^k \mathrm{d}w^k_t.$$

Since L_p norms are translation invariant, the assertion follows from (i).

The third assertion is a classical result (see [14]), and can be obtained directly from Iô's formula (see Lemma 4.3). □

Lemma 4.2. *Let* $u \in \mathcal{H}^1_{p,0}(T)$ *be a solution of Eq.* (4.1).

(i) *Take* $p_0 = p_0(d+1, \delta_0, K)$ *from* Lemma 4.1. *If* $p \in [2, p_0)$ *and* σ^i *are independent of* x, *then we have*

$$\|u_x\|_{\mathbb{L}_p(T)} \le N(\|u\|_{\mathbb{L}_p(T)} + \|f^i\|_{\mathbb{L}_p(T)} + \|f\|_{\mathbb{L}_p(T)} + \|g\|_{\mathbb{L}_p(T)}), \tag{4.4}$$

where $N = N(d, p, \delta_0, K)$ *(independent of* T*).*

(ii) (4.4) *holds if* $p = 2$.

Proof. (i) We use Agmon's idea and proceed as in [5]. Denote $\bar{f} = f + u$. Then u satisfies

$$\mathrm{d}u = [D_i(a^{ij}u_{x^i} + f^i) - u + \bar{f}]\mathrm{d}t + (\sigma^{ik}u_{x^i} + g^k)\mathrm{d}w^k_t.$$

Take an odd function $\eta \in C_0^\infty(\mathbb{R})$, $\eta \ne 0$. Consider the space $\mathbb{R}^{d+1} = \{(x,y) : x \in \mathbb{R}^d, y \in \mathbb{R}\}$, and define

$$\begin{aligned}
&\tilde{u}(t,z) = u(t,x)\eta(y)\cos(y), \qquad \tilde{g}(t,z) = \eta(y)\cos(y)g(t,x),\\
&\tilde{f}^i(t,z) = f^i(t,x)\eta(y)\cos(y) \quad \text{for } i = 1, \dots, d,\\
&\tilde{f}^{d+1}(t,z) = \bar{f}(t,x)\eta_1(y) + 2u(t,x)\eta_2(y) - u(t,x)\eta_3(y),
\end{aligned}$$

where

$$\begin{aligned}
&\eta_1(y) = \int_{-\infty}^y \eta(s)\cos(s)\mathrm{d}s, \quad \eta_3(y) = \int_{-\infty}^y \eta''(s)\cos(s)\mathrm{d}s,\\
&\eta_2(y) = \int_{-\infty}^y \eta'(s)\sin(s)\mathrm{d}s = -\eta'(y)\cos(y) + \eta_3(y).
\end{aligned}$$

Define $\tilde{a}^{ij}(t,z) = a^{ij}(t,x)$, $\tilde{\sigma}^{ik}(t,z) = \sigma^{ik}(t,x)$ if $0 \le i,j \le d$, $\tilde{a}^{i,j}(t,z) = 1$ if $i = j = d+1$, and $\tilde{a}^{ij}(t,z) = \tilde{\sigma}^{ik}(t,z) = 0$ otherwise. Then one can check that η_i are C^∞ functions with supports not wider than that of η, and $\tilde{u}$ satisfies

$$\mathrm{d}\tilde{u} = D_i(\tilde{a}^{ij}\tilde{u}_{z^j} + \tilde{f}^i)\mathrm{d}t + (\tilde{\sigma}^{ik}\tilde{u}_{z^i} + \tilde{g}^k)\mathrm{d}w^k_t.$$

Note that

$$\kappa := \int_{\mathbb{R}} |\eta(y)\cos(y)|^p \mathrm{d}y > 0.$$

By Lemma 4.1(ii), we get

$$\begin{aligned}\|u_x\|^p_{\mathbb{L}_p(T)} &= \kappa^{-1} E\int_0^T \int_{\mathbb{R}^{d+1}} |u_x \eta(y)\cos(y)|^p \mathrm{d}z\mathrm{d}t \\ &\le \kappa^{-1}\|\tilde{u}_z\|^p_{\mathbb{L}_p(T)} \le N\|\tilde{f^i}\|^p_{\mathbb{L}_p(T)} + N\|\tilde{g}\|^p_{\mathbb{L}_p(T)} \\ &\le N\|u\|^p_{\mathbb{L}_p(T)} + N\|f^i\|^p_{\mathbb{L}_p(T)} + N\|g\|^p_{\mathbb{L}_p(T)}.\end{aligned}$$

(ii) Just repeat the arguments in (i) using Lemma 4.1(iii). □

Lemma 4.3. *Let (2.6) hold and $u \in \mathcal{H}^1_{p,0}(T)$ be a solution of Eq. (4.1).*

(i) *There exists $p_1 = p_1(d, \delta_1, \delta_2, K) > 2$ such that if $p \in [2, p_1)$ and $f = 0$ then (4.3) holds with $N = N(d, p, \delta_i, K)$.*

(ii) *If $p \in [2, p_1)$ then (4.4) holds with $N = N(d, p, \delta_i, K)$.*

Proof. (i) We divide the proof into few steps.

Step 1. Assume that $\delta_0 > 1/2$. We will show that the constant $N(p)$ in Lemma 4.1(i) can be taken so that $N(2) = (2\delta_0 - 1)^{-1/2}$ and $\lim_{p\to 2+} N(p) = N(2)$.

Considering an approximation argument, without loss of generality we assume that u, a, f and g are sufficiently smooth in x. By Itô's formula

$$|u(T)|^2 = \int_0^T (2u(a^{ij}u_{x^j} + f^i)_{x^i} + |g|^2)\mathrm{d}t + \int_0^T 2ug^k \mathrm{d}w^k_t.$$

Take expectation, integrate over $\mathbb{R}^d$ to get

$$0 \le E\int_0^T \int_{\mathbb{R}^d} (-2a^{ij}u_{x^i}u_{x^j} - 2u_{x^i}f^i + |g|^2)\mathrm{d}x\mathrm{d}t.$$

Here

$$-2a^{ij}u_{x^i}u_{x^j} - 2u_{x^i}f^i \le -(2\delta_0 - 1)|u_x|^2 + |f^i|^2,$$

and consequently

$$(2\delta_0 - 1)E\int_0^T \int_{\mathbb{R}^d} |u_x|^2 \mathrm{d}x\mathrm{d}t \le E\int_0^T \int_{\mathbb{R}^d} (|f^i|^2 + |g|^2)\mathrm{d}x\mathrm{d}t.$$

This proves the assertion if $p = 2$. Indeed,

$$\begin{aligned}\|u_x\|_{\mathbb{L}_2(T)} &\le (2\delta_0 - 1)^{-1/2}(\|f^i\|^2_{\mathbb{L}_2(T)} + \|g\|^2_{\mathbb{L}_p(T)})^{1/2} \\ &\le (2\delta_0 - 1)^{-1/2}(\|f^i\|_{\mathbb{L}_2(T)} + \|g\|_{\mathbb{L}_p(T)}).\end{aligned}$$

Let $p \in [2, p_0)$ and consider the operator

$$\Phi : (f^i, g) \to Du,$$

where $u \in \mathcal{H}^1_{p,0}(T)$ is the solution of Eq. (4.1) with $f = \sigma^i = 0$. Then by the (real-valued version) Riesz–Thorin theorem for any $r \in (2, p_0)$ and $p \in [2, r]$,

$$\|\Phi\|_p \le \|\Phi\|_2^{1-\alpha}\|\Phi\|_r^{\alpha}, \quad \alpha = (1/2 - 1/p)/(1/2 - 1/r).$$

Since we can define $\|\Phi\|_2^{1-\alpha}\|\Phi\|_r^{\alpha}$ as a new $N(p)$ we may assume that $N(p)$ is continuous in $p \in [2, r)$. Therefore

$$\lim_{p\to 2+} N(p) = \lim_{p\to 2+} \|\Phi\|_2^{1-\alpha}\|\Phi\|_r^{\alpha} = \|\Phi\|_2.$$

Step 2. Let $\delta_0 > 1/2$. Then

$$0 < \frac{2\delta_1}{2(\delta_0+\delta_1)-1} < 1. \tag{4.5}$$

We will show that there exists $\bar{p}_1 > 2$ such that the assertion holds if $p \in [2, \bar{p}_1)$. By (2.6),

$$(\delta_0+\delta_1)I < (a^{ij}) < KI.$$

Take $p_0 = p_0(d, \delta_0+\delta_1, K)$ from Lemma 4.1(i). Then by Lemma 4.1(i), if $p \in [2, p_0)$ then

$$\begin{aligned}\|u_x\|_{\mathbb{L}_p(T)} &\le N(p)\|f^i\|_{\mathbb{L}_p(T)} + N(p)\|\sigma^i u_{x^i} + g\|_{\mathbb{L}_p(T)}\\ &\le N(p)\|f^i\|_{\mathbb{L}_p(T)} + N(p)\|\sigma^i u_{x^i}\|_{\mathbb{L}_p(T)} + N(p)\|g\|_{\mathbb{L}_p(T)},\end{aligned}$$

where, by Step 1, $\lim_{p\to 2+} N(p) = (2(\delta_0+\delta_1)-1)^{-1/2}$. Here

$$|\sigma^i u_{x^i}|_{\ell_2} = \sqrt{2\alpha^{ij}u_{x^i}u_{x^j}} \le \sqrt{2\delta_1}|u_x|.$$

Consequently,

$$\|u_x\|_{\mathbb{L}_p(T)} \le N(p)\sqrt{2\delta_1}\|u_x\|_{\mathbb{L}_p(T)} + N\|f^i\|_{\mathbb{L}_p(T)} + N\|g\|_{\mathbb{L}_p(T)}.$$

It remains to observe that

$$\lim_{p\downarrow 2} N(p)\sqrt{2\delta_1} = \left(\frac{2\delta_1}{2(\delta_0+\delta_1)-1}\right)^{1/2} < 1.$$

Step 3. General case. Fix $c \in (0,1)$ such that $\delta_{0c} := c^{-1}\delta_0 > 1/2$. Define $\delta_{1c} := c^{-1}\delta_1$ and

$$v(t,x) = u(t,\sqrt{c}x), \quad a_c(t,x) = c^{-1}a(t,\sqrt{c}x), \quad f_c^i(t,x) = c^{-1/2}f^i(t,\sqrt{c}x),$$
$$g_c(t,x) = g(t,\sqrt{c}x), \quad \sigma_c(t,x) = c^{-1/2}\sigma(t,\sqrt{c}x), \quad \alpha_c^{ij} = \frac{1}{2}\sigma_c^{ik}\sigma_c^{jk}.$$

Then v satisfies

$$\mathrm{d}v = D_i(a_c^{ij}v_{x^j} + f_c^i)\mathrm{d}t + (\sigma_c^{ik}v_{x^i} + g_c^k)\mathrm{d}w_t^k$$

and

$$(\alpha_c^{ij}) < \delta_{1c}I < (\delta_{0c}+\delta_{1c})I < (a_c^{ij}) < Kc^{-1}I.$$

By Step 2, there exists $p_1 = p_1(d, c^{-1}\delta_i, K) > 2$ such that if $p \in [2, p_1)$ then

$$\|v_x\|_{\mathbb{L}_p(T)} \le N\|f_c^i\|_{\mathbb{L}_p(T)} + N\|g_c\|_{\mathbb{L}_p(T)}.$$

Consequently

$$\|u_x\|_{\mathbb{L}_p(T)} \le N\|f^i\|_{\mathbb{L}_p(T)} + N\|g\|_{\mathbb{L}_p(T)},$$

where N is independent of T.

(ii) Just repeat the proof of Lemma 4.2(i). □

Proof of Theorem 2.9. Again we only prove a priori estimate (2.5) given that $p \in [2, p_1)$, $\tau \equiv T$, $u_0 = 0$ and $b^i = 0$. By Theorem 5.1 in [8] the equation

$$\mathrm{d}v = (\Delta v + (\bar{b}^i u)_{x^i} + cu + f)\mathrm{d}t + \nu^k u \mathrm{d}w_t^k$$

has a solution $v \in \mathcal{H}_{p,0}^1(T)$, and for any $t \leq T$

$$\|v\|_{\mathbb{H}_p^1(t)} \leq N(d, p, T)(\|(\bar{b}^i u)_{x^i} + cu + f\|_{\mathbb{H}_p^{-1}(t)} + \|\nu u\|_{\mathbb{L}_p(t)}).$$

Since $D : H_p^\gamma \to H_p^{\gamma-1}$ is a bounded operator and $\|\cdot\|_{H_p^{\gamma-1}} \leq \|\cdot\|_{H_p^\gamma}$,

$$\|v\|_{\mathbb{H}_p^1(t)}^p \leq N\|u\|_{\mathbb{L}_p(t)}^p + N\|f\|_{\mathbb{H}_p^{-1}(t)}^p. \tag{4.6}$$

Note that $\bar{u} := u - v \in \mathcal{H}_{p,0}^1(T)$ and satisfies

$$\mathrm{d}\bar{u} = D_i(a^{ij}\bar{u}_{x^j} + \bar{f}^i)\mathrm{d}t + (\sigma^{ik}\bar{u}_{x^i} + \bar{g}^k)\mathrm{d}w_t^k,$$

where $\bar{f}^i := f^i + (a^{ij} - \delta^{ij})v_{x^j}$ and $\bar{g}^k := g^k + \sigma^{ik}v_{x^i}$.

Using Lemma 4.3 and (4.6), we easily get

$$\|u\|_{\mathbb{H}_p^1(t)}^p \leq N\|u\|_{\mathbb{L}_p(t)}^p + N\|f^i\|_{\mathbb{L}_p(t)}^p + N\|f\|_{\mathbb{H}_p^{-1}(t)}^p + \|g\|_{\mathbb{L}_p(t)}^p.$$

This and (3.8) yield

$$\|u\|_{\mathcal{H}_p^1(t)} \leq N\int_0^t \|u\|_{\mathcal{H}_p^1(s)}^p \mathrm{d}s + N\|f^i\|_{\mathbb{L}_p(T)}^p + N\|f\|_{\mathbb{H}_p^{-1}(T)}^p + N\|g\|_{\mathbb{L}_p(T)}^p.$$

Thus (2.5) follows from Gronwall's inequality.

Proof of Theorem 2.8. First assume that σ^i are independent of x. In this case, it suffices to repeat the proof of Theorem 2.9 using Lemma 4.2 instead of Lemma 4.3. The general case is handled using standard arguments such as a partition of unity and perturbation argument. We only refer to the proof of Theorem 5.1 in [8]. The theorem is proved.

5. SPDEs on $\mathbb{R}_+^d$

In this section we develop some estimates for SPDEs defined on $\mathbb{R}_+^d$. By M^α we denote the operator of multiplying $(x^1)^\alpha$ and $M = M^1$.

Remember that we write $u \in \mathfrak{H}_{p,\theta}^{\gamma,q}(\tau)$ if $u \in M\mathbb{H}_{p,\theta}^{\gamma,q}(\tau), u(0,\cdot) \in U_{p,\theta}^{\gamma,q}$ and for some $f \in M^{-1}\mathbb{H}_{p,\theta}^{\gamma-2,q}(\tau)$, $g \in \mathbb{H}_{p,\theta}^{\gamma-1,q}(\tau)$

$$\mathrm{d}u = f\mathrm{d}t + g^k\mathrm{d}w_t^k$$

in the sense of distribution.

First we extend Theorems 3.1 and 3.2 in [7] as follows.

Lemma 5.1. *Let (2.1) hold, and a^{ij} and σ^{ik} be independent of x. Then there exists $\kappa_0 \in (0, 1)$ such that if*

$$d - \kappa_0 \leq \theta < d - 1 + p \tag{5.1}$$

then for any $f \in M^{-1}\mathbb{H}^{\gamma,q}_{p,\theta}(T)$, $g \in \mathbb{H}^{\gamma+1,q}_{p,\theta}(T)$ *and* $M^{2/q-1-\varepsilon_0}u_0 \in L_q(\Omega, \mathcal{F}_0, H^{\gamma+2-2/q+\varepsilon_0}_{p,\theta})$, *the equation*

$$du = (a^{ij}u_{x^ix^j} + f)dt + (\sigma^{ik}u_{x^i} + g^k)dw^k_t$$

with initial data u_0 *has a unique solution* $u \in M\mathbb{H}^{\gamma+2,q}_{p,\theta}(T)$, *and for this solution*

$$\|M^{-1}u\|^q_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(T)} \leq N\|Mf\|^q_{\mathbb{H}^{\gamma,q}_{p,\theta}(T)} + N\|g\|^q_{\mathbb{H}^{\gamma+1,q}_{p,\theta}(T)} + NE\|M^{2/q-1-\varepsilon_0}u_0\|^q_{H^{\gamma+2-2/q+\varepsilon_0}_{p,\theta}}. \tag{5.2}$$

Proof. We know from Theorem 3.2 in [7] that the lemma holds if $\sigma^{ik} = 0$. Thus we only show that there exists $\kappa_0 > 0$ such that a priori estimate (5.2) holds given that (5.1) holds and $u_0 = 0$.

By Theorem 3.1 in [7], there exists $\varepsilon_1 > 0$ such that Lemma 5.1 holds for any $\theta \in (d-1+p-\varepsilon_1, d-1+p)$. Also Lemma 5.1 and the proof of Theorem 3.2 in [7] show that if the claim of the lemma holds when $\theta = \theta_1$ and $\theta = \theta_2$, where $\theta_1 \leq \theta_2$, then there exists $\kappa_0 > 0$ such that the lemma also holds for any $\theta \in [\theta_1 - \kappa_0, \theta_2 + \kappa_0]$. Furthermore the results for $\gamma \neq 0$ directly follows from the result for $\gamma = 0$ (see Lemma 4.1 and proof of Lemma 4.3 in [7]). Thus to prove (5.2) we only need to consider the case $\theta = d$ and $\gamma = 0$. It is well know that the set

$$\cup_{k=1}^{\infty} L_q(\Omega, C([0,T], C^n_0(\mathcal{O}_k))),$$

where $\mathcal{O}_k = (1/k, k) \times \{|x'| < k\}$, is everywhere dense in $\mathfrak{H}^{\gamma+2,q}_{p,\theta}(T)$. Thus without loss of generality we assume that u is sufficiently smooth in x and vanishes outside some $\mathcal{O}_k$. Consequently $u \in \mathcal{H}^{1,q}_p(T)$.

By Lemma 4.1 in [7],

$$\|M^{-1}u\|_{\mathbb{H}^{2,q}_{p,d}(T)} \leq N\|Mf\|_{\mathbb{L}^q_{p,d}(T)} + N\|g\|_{\mathbb{H}^{1,q}_{p,d}(T)} + N\|M^{-1}u\|_{\mathbb{H}^{1,q}_{p,d}(T)}.$$

Thus we only need to estimate $\|M^{-1}u\|_{\mathbb{H}^{1,q}_{p,d}(T)}$. By Corollary 2.12 of [9] we have the representation $f = D_i\bar{f}^i$, where $\bar{f}^i \in \mathbb{L}^q_{p,d}(T)$ and

$$\sum_{i=1}^{d} \|\bar{f}^i\|_{\mathbb{L}^q_{p,d}(T)} \leq N\|Mf\|_{\mathbb{H}^{-1,q}_{p,d}(T)}.$$

It follows from Lemma 2.12(iii) and Theorem 2.5 that

$$\begin{aligned}\|M^{-1}u\|_{\mathbb{H}^{1,q}_{p,d}(T)} &\leq N\|u_x\|_{\mathbb{L}^q_{p,d}(T)} = N\|u_x\|_{\mathbb{L}^q_p(T)} \leq N\|\bar{f}^i\|_{\mathbb{L}^q_p(T)} + N\|g\|_{\mathbb{L}^q_p(T)} \\ &= N\|\bar{f}^i\|_{\mathbb{L}^q_{p,d}(T)} + N\|g\|_{\mathbb{L}^q_{p,d}(T)} \leq N\|Mf\|_{\mathbb{H}^{-1,q}_{p,d}(T)} + N\|g\|_{\mathbb{L}^q_{p,d}(T)}.\end{aligned}$$

This finishes the proof of the lemma. □

Theorem 5.2. *Let* (2.1), *Assumption* 2.16 *and* (5.1) *hold, and for each* ω, t, x, y,

$$|a^{ij}(t,x) - a^{ij}(t,y)| + |\sigma^i(t,x) - \sigma^i(t,y)|_{\ell_2} + x^1|b^i(t,x)| + (x^1)^2|c(t,x)| + (x^1)|\nu(t,x)|_{\ell_2} < \beta. \tag{5.3}$$

Then there exists $\beta_0 = \beta_0(d, p, q, \gamma, \delta, K) \in (0, 1)$ *such that if* $\beta \leq \beta_0$ *then for any* $f \in M^{-1}\mathbb{H}^{\gamma,q}_{p,\theta}(T)$*,* $g \in \mathbb{H}^{\gamma+1,q}_{p,\theta}(T)$ *and* $M^{2/q-1-\varepsilon_0}u_0 \in L_q(\Omega, \mathcal{F}_0, H^{\gamma+2-2/q+\varepsilon_0}_{p,\theta})$*, Eq.* (1.1) *with initial data* u_0 *admits a unique solution* $u \in M\mathbb{H}^{\gamma+2,q}_{p,\theta}(T)$*, and for this solution*

$$\|M^{-1}u\|^q_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(T)} \leq N\|Mf\|^q_{\mathbb{H}^{\gamma,q}_{p,\theta}(T)} + N\|g\|^q_{\mathbb{H}^{\gamma+1,q}_{p,\theta}(T)} + NE\|M^{2/q-1-\varepsilon_0}u_0\|^q_{H^{\gamma+2-2/q+\varepsilon_0}_{p,\theta}}, \tag{5.4}$$

where $N = N(d, p, q, \gamma, \varepsilon, \delta, K)$*.*

Proof. Here we prove the theorem only when $|\gamma| \notin \{1, 2, \ldots\}$. The result when $\gamma = 0$ yields the result when γ is an integer (see [3], where the theorem is proved when $p = q$). We show that there exists β_0 such that the estimate (5.4) holds given that a solution u already exists and $\beta \leq \beta_0$, $u_0 = 0$.

Case 1. Assume that γ is not an integer. Fix $x_0 \in \mathbb{R}^d_+$ and denote $a_0(t, x) = a(t, x_0)$, $\sigma_0(t, x) = \sigma(t, x_0)$. Note that u satisfies

$$\mathrm{d}u = (a^{ij}_0 u_{x^i x^j} + f_0)\mathrm{d}t + (\sigma^{ik}_0 u_{x^i} + g^k_0)\mathrm{d}w^k_t$$

where

$$f_0 = (a^{ij} - a^{ij}_0)u_{x^i x^j} + b^i u_{x^i} + cu + f, \qquad g^k_0 = (\sigma^{ik} - \sigma^{ik}_0)u_{x^i} + \nu^k u + g^k.$$

By Lemma 5.1,

$$\|M^{-1}u\|_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(T)} \leq N\|Mf_0\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(T)} + N\|g_0\|_{\mathbb{H}^{\gamma+1,q}_{p,\theta}(T)}. \tag{5.5}$$

Remember that

$$\|Mu_{xx}\|_{H^{\gamma}_{p,\theta}} + \|u_x\|_{H^{\gamma}_{p,\theta}} \leq N\|M^{-1}u\|_{H^{\gamma+2}_{p,\theta}}.$$

Thus by Lemma 3.6 in [4]

$$\begin{aligned}\|M(a^{ij} - a^{ij}_0)u_{x^i x^j}\|_{H^{\gamma}_{p,\theta}} &\leq N \sup_{\omega,t,x} |a^{ij}(t, x) - a^{ij}_0(t, x)|^s \, \|Mu_{xx}\|_{H^{\gamma}_{p,\theta}} \\ &\leq N\beta^s\|M^{-1}u\|_{H^{\gamma+2}_{p,\theta}},\end{aligned}$$

where $s := 1 - \frac{|\gamma|}{|\gamma|_+} > 0$, since γ is not integer. Similarly

$$\|(\sigma^{ik} - \sigma^{ik}_0)u_{x^i}\|_{H^{\gamma+1}_{p,\theta}} \leq N\beta^s\|u_x\|_{H^{\gamma+1}_{p,\theta}} \leq N\beta^s\|M^{-1}u\|_{H^{\gamma+2}_{p,\theta}},$$

$$\|Mbu_x\|_{H^{\gamma}_{p,\theta}} + \|Mcu\|_{H^{\gamma}_{p,\theta}} + \|\nu u\|_{H^{\gamma+1}_{p,\theta}} \leq N\beta^s\|u_x\|_{H^{\gamma}_{p,\theta}} + N\beta^s\|M^{-1}u\|_{H^{\gamma+1}_{p,\theta}}.$$

Consequently,

$$\|M^{-1}u\|_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(T)} \leq N\beta^s\|M^{-1}u\|_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(T)} + N\|Mf\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(T)} + N\|g\|_{\mathbb{H}^{\gamma+1,q}_{p,\theta}(T)}.$$

Now it suffices to take β_0 sufficiently small such that $N\beta^s_0 \leq 1/2$.

Case 2. Let $\gamma = 0$. Proceed as before and reach (5.5). Obviously,

$$\|M(a^{ij} - a^{ij}_0)u_{x^i x^j}\|_{L_{p,\theta}} + \|Mbu_x\|_{L_{p,\theta}} + \|Mcu\|_{L_{p,\theta}}$$

$$\le \sup_{\omega,t,x} |a^{ij}(t,x) - a_0^{ij}(t,x)| \, \|Mu_{xx}\|_{L_{p,\theta}} + \sup_{\omega,t,x} |Mb^i(t,x)| \|u_x\|_{L_{p,\theta}}$$
$$+ \sup_{\omega,t,x} |M^2 c(t,x)| \|M^{-1}u\|_{L_{p,\theta}} \le N\beta \|M^{-1}u\|_{H^2_{p,\theta}}.$$

Also by Lemma 3.6 in [4]

$$\|(\sigma^{ik} - \sigma_0^{ik})u_{x^i}\|_{H^1_{p,\theta}} \le N \sup_{\omega,t,x} |\sigma^{ik} - \sigma_0^{ik}| \, \|u_x\|_{H^1_{p,\theta}}$$
$$+ N \sup_{\omega,t} |\sigma^i(t,\cdot)|_1^{(0)} \, \|u_x\|_{L_{p,\theta}} \le N\beta \|M^{-1}u\|_{H^2_{p,\theta}} + N\|M^{-1}u\|_{H^1_{p,\theta}}.$$

Similarly

$$\|\nu u\|_{H^1_{p,\theta}} \le N\beta \|M^{-1}u\|_{H^1_{p,\theta}} + N\|M^{-1}u\|_{L_{p,\theta}}.$$

Thus if β is sufficiently small,

$$\|M^{-1}u\|_{\mathbb{H}^{2,q}_{p,\theta}(T)} \le N\|M^{-1}u\|_{\mathbb{H}^{1,q}_{p,\theta}(T)} + N\|Mf\|_{\mathbb{L}^q_{p,\theta}(T)} + N\|g\|_{\mathbb{H}^{1,q}_{p,\theta}(T)}. \tag{5.6}$$

To estimate $\|M^{-1}u\|_{\mathbb{H}^{1,q}_{p,\theta}(T)}$ we use (5.5) with $\gamma = -1$, and get (remember that $\|\cdot\|_{H^{-1}_{p,\theta}} \le \|\cdot\|_{L_{p,\theta}}$)

$$\|M^{-1}u\|_{\mathbb{H}^{1,q}_{p,\theta}(T)} \le N\|Mf_0\|_{\mathbb{L}_{p,\theta}(T)} + N\|g_0\|_{\mathbb{L}^q_{p,\theta}(T)}.$$

As is easy to check

$$\|Mf_0\|_{\mathbb{L}^q_{p,\theta}(T)} + \|g_0\|_{\mathbb{L}^q_{p,\theta}(T)} \le N\beta\|M^{-1}u\|_{\mathbb{H}^{2,q}_{p,\theta}(T)} + N\|Mf\|_{\mathbb{L}^q_{p,\theta}(T)} + N\|g\|_{\mathbb{L}^q_{p,\theta}(T)}.$$

This and (5.6) finish the proof. □

Theorem 5.3. *Let (2.1) hold. Also assume that (5.1) and (5.3) are satisfied. Then there exists $\beta_1 > 0$ such that if $\beta \le \beta_1$ then for any $f^i \in \mathbb{L}^q_{p,\theta}(T)$, $f \in M^{-1}\mathbb{H}^{-1,q}_{p,\theta}(T)$, $g \in \mathbb{L}^q_{p,\theta}(T)$ and $M^{2/q-1-\varepsilon_0}u_0 \in L_q(\Omega, \mathcal{F}_0, H^{1-2/q+\varepsilon_0}_{p,\theta})$, equation (1.2) with initial data u_0 has a unique solution $u \in M\mathbb{H}^{1,q}_{p,\theta}(T)$, and for this solution*

$$\|M^{-1}u\|^q_{\mathbb{H}^{1,q}_{p,\theta}(T)} \le N\|f^i\|^q_{\mathbb{L}^q_{p,\theta}(T)} + N\|Mf\|^q_{\mathbb{H}^{-1,q}_{p,\theta}(T)}$$
$$+ N\|g\|^q_{\mathbb{L}^q_{p,\theta}(T)} + NE\|M^{2/q-1-\varepsilon_0}u_0\|^q_{H^{1-2/q+\varepsilon_0}_{p,\theta}}, \tag{5.7}$$

where $N = N(d, p, q, \gamma, \varepsilon, \delta, K)$.

Proof. Again we assume that $u_0 = 0$. We proceed as in the proof Theorem 5.2. Denote

$$f_0 = D_i((a^{ij} - a_0^{ij})u_{x^j} + \bar{b}^i u + f^i) + b^i u_{x^i} + cu + f,$$
$$g_0^k = (\sigma^{ik} - \sigma_0^{ik})u_{x^i} + \nu^k u + g^k.$$

Then by Lemma 5.1,

$$\|M^{-1}u\|_{\mathbb{H}^{1,q}_{p,\theta}(T)} \le M\|Mf_0\|_{\mathbb{H}^{-1,q}_{p,\theta}(T)} + N\|g_0\|_{\mathbb{L}^q_{p,\theta}(T)}.$$

Since $MD : H^{\gamma}_{p,\theta} \to H^{\gamma-1}_{p,\theta}$ is a bounded operator, one easily gets

$$\|Mf_0\|_{\mathbb{H}^{-1,q}_{p,\theta}(T)} + \|g_0\|_{\mathbb{L}^q_{p,\theta}(T)} \le N\beta\|M^{-1}u\|_{\mathbb{H}^{1,q}_{p,\theta}(T)} + N\|f^i\|_{\mathbb{L}^q_{p,\theta}(T)} + \|Mf\|_{\mathbb{H}^{-1,q}_{p,\theta}(T)} + N\|g\|_{\mathbb{L}^q_{p,\theta}(T)}.$$

Obviously, this proves the theorem. □

Theorems 2.19–2.21 are based on the following result.

Theorem 5.4. *Let (2.4) hold,*

$$|M\bar{b}^i(t,x)| + |Mb^i(t,x)| + |M^2c(t,x)| + |M\nu(t,x)|_{\ell_2} \le \beta \quad \forall \omega, t, x \tag{5.8}$$

and one of the following three conditions be satisfied:

(a) $p = 2$,
(b) $p \in [2, p_0)$ *and* $|\sigma^i(t,x) - \sigma^i(t,y)| \le \beta$, $\forall \omega, t, x, y$,
(c) (2.6) *holds and* $p \in [2, p_1)$.

Then there exist constants $\beta_0, \kappa_1 > 0$, *depending only on* d, p, δ_i *and* K *such that if* $\beta \le \beta_0$ *and* (2.19) *holds, then for any* $f^i \in \mathbb{L}_{p,\theta}(T)$, $g \in \mathbb{L}_{p,\theta}(T)$, $f \in M^{-1}\mathbb{H}^{-1}_{p,\theta}(T)$ *and* $u_0 \in U^1_{p,\theta}$ *Eq.* (1.2) *has a unique solution* $u \in \mathfrak{H}^1_{p,\theta}(T)$, *and*

$$\|u\|_{\mathfrak{H}^1_{p,\theta}(T)} \le N(\|f^i\|_{\mathbb{L}_{p,\theta}(T)} + \|Mf\|_{\mathbb{H}^{-1}_{p,\theta}(T)} + \|g\|_{\mathbb{L}_{p,\theta}(T)} + \|u_0\|_{U^1_{p,\theta}}), \tag{5.9}$$

where $N = N(d, p, \theta, \delta_i, K, T)$

We need the following lemmas to prove Theorem 5.4.

Lemma 5.5. *For any* $\varepsilon > 0$ *there exists* $\zeta_\varepsilon \in C_0^\infty(1,\infty)$ *such that*

$$x^{\theta-d} \le \sum_{n=-\infty}^{\infty} \mathrm{e}^{n(\theta-d)}|\zeta_\varepsilon(\mathrm{e}^{-n}x)|^p \le N(\theta,d)x^{\theta-d}, \quad \forall x \in \mathbb{R}_+ \tag{5.10}$$

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{n(\theta-d)}|\zeta_{\varepsilon x}(\mathrm{e}^{-n}x)|^p \le \varepsilon x^{\theta-d}, \quad \forall x \in \mathbb{R}_+ \tag{5.11}$$

Consequently,

$$\int_{\mathbb{R}^d_+} |u(x)|^p (x^1)^{\theta-d}\mathrm{d}x \le \sum_{n=-\infty}^{\infty} \mathrm{e}^{n\theta}\|u(\mathrm{e}^n\cdot)\zeta_\varepsilon(\cdot)\|^p_{L_p} \le N(\theta,d)\int_{\mathbb{R}^d_+} |u(x)|^p(x^1)^{\theta-d}\mathrm{d}x,$$

and

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{n\theta}\|u(\mathrm{e}^n\cdot)\zeta_{\varepsilon x}(\cdot)\|^p_{L_p} \le \varepsilon\int_{\mathbb{R}^d_+} |u(x)|^p(x^1)^{\theta-d}\mathrm{d}x.$$

Proof. The function

$$p_0(t) := \sum_{n=-\infty}^{\infty} \mathrm{e}^{(n-t)(\theta-d)} \zeta^p(\mathrm{e}^{t-n})$$

is obviously periodic with period 1. It follows that it is bounded and bounded away from zero. Thus (by multiplying a proper constant to $p(t)$ if necessary) we get

$$1 \le p(t) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{(n-t)(\theta-d)} \zeta^p(\mathrm{e}^{t-n}) \le N(\theta, d), \quad \forall t.$$

Observe

$$p(\ln x)x^{\theta-d} = \sum_{n=-\infty}^{\infty} \mathrm{e}^{n(\theta-d)} \zeta^p(\mathrm{e}^{-n}x), \quad \forall x \in \mathbb{R}_+.$$

It follows that

$$x^{\theta-d} \le \sum_{n=-\infty}^{\infty} \mathrm{e}^{n(\theta-d)} \zeta^p(\mathrm{e}^{-n}x) \le N(\theta, d)x^{\theta-d}, \quad \forall x \in \mathbb{R}_+.$$

Similarly

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{n(\theta-d)} |\zeta_x(\mathrm{e}^{-n}x)|^p \le N x^{\theta-d}, \quad \forall x \in \mathbb{R}_+.$$

To get (5.10) and (5.11) it is enough to take $\zeta_\varepsilon(x) = (\frac{\varepsilon}{N})^{(d-\theta)/p}\zeta(\varepsilon x/N)$. For the second assertion of the lemma, observe that for $h = \zeta_\varepsilon$ or $h = \zeta_{\varepsilon x}$

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{n\theta} \|u(\mathrm{e}^n \cdot)h\|^p_{L_p} = \int_{\mathbb{R}^d_+} |u|^p(x) \left[\sum_{n=-\infty}^{\infty} \mathrm{e}^{n(\theta-d)} |h(\mathrm{e}^{-n}x)|^p\right] \mathrm{d}x.$$

The lemma is proved. $\square$

Lemma 5.6. *Let (2.4) hold with one of the following conditions:*

(a) $p = 2$
(b) $p \in [2, p_0)$ *and* $\sigma^i(t, x) = \sigma^i(t)$
(c) (2.6) *holds and* $p \in [2, p_1)$

Then if $u \in \mathfrak{H}^1_{p,\theta,0}(T)$ *is a solution of* (4.1) *with* $f = 0$, *then*

$$\|u_x\|_{\mathbb{L}_{p,\theta}(T)} \le N\|M^{-1}u\|_{\mathbb{L}_{p,\theta}(T)} + N\|f^i\|_{\mathbb{L}_{p,\theta}(T)} + N\|g\|_{\mathbb{L}_{p,\theta}(T)}. \tag{5.12}$$

Proof. Take $\varepsilon \in (0, 1)$ to be specified later and take ζ_ε from Lemma 5.5. By (2.9) and (2.10) and Lemma 5.5,

$$\|u_x\|^p_{\mathbb{L}_{p,\theta}(T)} \le NE\int_0^T \int_{\mathbb{R}^d_+} |u_x|^p (x^1)^{\theta-d} \mathrm{d}x\mathrm{d}t$$
$$\le N \sum_{n=-\infty}^{\infty} \mathrm{e}^{n\theta} \|u_x(\mathrm{e}^n \cdot)\zeta_\varepsilon\|^p_{\mathbb{L}_p(T)}$$

$$
\begin{aligned}
&\le N \sum_{n=-\infty}^{\infty} e^{n(\theta-p)} (\|(u(e^n \cdot)\zeta_\varepsilon)_x\|^p_{L_p(T)} + \|u(e^n \cdot)\zeta_{\varepsilon x}\|^p_{\mathbb{L}_p(T)}) \\
&= N \sum_{n=-\infty}^{\infty} e^{n(\theta-p)} \|(u(e^n \cdot)\zeta_\varepsilon)_x\|^p_{L_p(T)} + N \sum_{n=-\infty}^{\infty} e^{n\theta} \|(M^{-1}u)(e^n \cdot) M\zeta_{\varepsilon x}\|^p_{\mathbb{L}_p(T)} \\
&\le N \sum_{n=-\infty}^{\infty} e^{n(\theta-p+2)} \|(u(e^{2n}\cdot, e^n \cdot)\zeta_\varepsilon)_x\|^p_{L_p(e^{-2n}T)} + N(\varepsilon)\|M^{-1}u\|^p_{\mathbb{L}_{p,\theta}(T)},
\end{aligned} \tag{5.13}
$$

where N is independent of ε.

Denote $u_n(t,x) := u(e^{2n}t, e^n x)\zeta_\varepsilon(x)$. Then $u_n \in \mathcal{H}^1_{p,0}(e^{-2n}T)$ satisfies

$$
du_n = (D_i(a^{ij}_n u_{nx^j} + f^i_n) + f_n)dt + (\sigma^{ik}_n u_{nx^i} + g^k_n)dw^k_t(n)
$$

where $w^k_t(n) := e^{-n} w^k_{e^{2n}t}$, $k = 1, 2, \dots$, are independent one dimensional Wiener processes and

$$
\begin{aligned}
&a^{ij}_n(t,x) := a^{ij}(e^{2n}t, e^n x), \qquad \sigma^{ik}_n(t,x) := \sigma^{ik}(e^{2n}t, e^n x), \\
&f^i_n(t,x) := e^n f^i(e^{2n}t, e^n x)\zeta_\varepsilon(x) - a^{ij}_n(t,x) u(e^{2n}t, e^n x)\zeta_{\varepsilon x^i}(x), \\
&f_n(t,x) := -a^{ij}_n(t,x) e^n u_{x^i}(e^{2n}t, e^n x)\zeta_{\varepsilon x^i}(x) - e^n f^i(e^{2n}t, e^n x)\zeta_{\varepsilon x^i}(x), \\
&g^k_n(t,x) := e^n g^k(e^{2n}t, e^n x)\zeta_\varepsilon(x) - \sigma^{ik}_n(t,x) u(e^{2n}t, e^n x)\zeta_{\varepsilon x^i}(x).
\end{aligned}
$$

By Lemmas 4.2 and 4.3,

$$
\begin{aligned}
\|u_{nx}\|_{\mathbb{L}_p(e^{-2n}T)} \le{}& N\|u_n\|_{\mathbb{L}_p(e^{-2n}T)} + N\|f^i_n\|_{\mathbb{L}_p(e^{-2n}T)} \\
&+ N\|f_n\|_{\mathbb{L}_p(e^{-2n}T)} + N\|g_n\|_{\mathbb{L}_p(e^{-2n}T)},
\end{aligned} \tag{5.14}
$$

where N is independent of n and T. Coming back to (5.13) we get

$$
\begin{aligned}
\|u_x\|^p_{\mathbb{L}_{p,\theta}(T)} \le{}& N(\varepsilon)\|M^{-1}u\|^p_{\mathbb{L}_{p,\theta}(T)} + N\sum e^{n\theta}\|f^i(\cdot, e^n \cdot)\zeta_\varepsilon\|^p_{\mathbb{L}_p(T)} \\
&+ N\sum e^{n\theta}\|(M^{-1}u)(\cdot, e^n \cdot)M\zeta_{\varepsilon x}\|^p_{\mathbb{L}_p(T)} + N\sum e^{n\theta}\|f^i(\cdot, e^n \cdot)\zeta_{\varepsilon x}\|^p_{\mathbb{L}_p(T)} \\
&+ N\sum e^{n\theta}\|g(\cdot, e^n \cdot)\zeta_\varepsilon\|^p_{\mathbb{L}_p(T)} + N\sum e^{n\theta}\|u_x(\cdot, e^n \cdot)\zeta_{\varepsilon x}\|^p_{\mathbb{L}_p(T)},
\end{aligned}
$$

where N is independent of ε. By (2.10) and Lemma 5.5, to get (5.12), it is enough to take $\varepsilon > 0$ such that $N\varepsilon \le 1/2$. The lemma is proved. □

Lemma 5.7. *Let (2.4) hold with one of (a)–(c) in Lemma 5.6. Let $\theta = d$ or*

$$
d - 2 + p - \frac{1}{\frac{Kp}{\delta_0(p-1)} - 1} < \theta < d - 2 + p + \frac{1}{\frac{Kp}{\delta_0(p-1)} + 1}. \tag{5.15}
$$

Suppose that $u \in \mathfrak{H}^1_{p,\theta,0}(T)$ is a solution of (4.1) with $f = 0$ such that u is sufficiently smooth in x and has a compact support in $\mathbb{R}^d_+$. Then there exists a constant N depending only on θ, d, p such that

$$
\|M^{-1}u\|_{\mathbb{L}_{p,\theta}(T)} \le N\|f^i\|_{\mathbb{L}_{p,\theta}(T)} + N\|g\|_{\mathbb{L}_{p,\theta}(T)}. \tag{5.16}
$$

Proof. By applying Iô's formula to $(x^1)^c|u(t,x)|^p$ with $c = 2+\theta-d-p$,

$$|u(t)|^p(x^1)^c = \int_0^t [p|u|^{p-2}uD_i(a^{ij}u_{x^j} + f^i) + \frac{1}{2}p(p-1)|\sigma^{ik}u_{x^i} + g^k|^2_{\ell_2}](x^1)^c\mathrm{d}t$$
$$+ \int_0^t p|u|^{p-2}u(\sigma^{ik}u_{x^i} + g^k)(x^1)^c\mathrm{d}w^k_t.$$

Taking the expectation and integrating with respect to x to get

$$0 \leq E\int_0^t \int_{\mathbb{R}^d_+} -p(p-1)|u|^{p-2}(a^{ij}u_{x^i}u_{x^j} + u_{x^i}f^i)(x^1)^c\mathrm{d}x\mathrm{d}t$$
$$+ E\int_0^t \int_{\mathbb{R}^d_+} -p|u|^{p-2}u(a^{ij}u_{x^j} + f^i)c(x^1)^{c-1}\mathrm{d}x\mathrm{d}t$$
$$+ E\int_0^t \int_{\mathbb{R}^d_+} \frac{1}{2}p(p-1)|u|^{p-2}|\sigma^{ik}u_{x^i} + g^k|^2_{\ell_2}(x^1)^c\mathrm{d}x\mathrm{d}t.$$

As is easy to check

$$p(p-1)|u|^{p-2}\left(-a^{ij}u_{x^i}u_{x^j} + \frac{1}{2}|\sigma^{ik}u_{x^i} + g^k|^2_{\ell_2}\right)(x^1)^c$$
$$\leq -(\delta_0-\delta)p(p-1)|u|^{p-2}|u_x|^2(x^1)^c + N(\delta)|M^{-1}u|^{p-2}|g|^2(x^1)^{\theta-d}.$$

Also

$$-p(p-1)|u|^{p-2}u_{x^i}f^i(x^1)^c \leq \delta|u|^{p-2}|u_x|^2(x^1)^c + N|u|^{p-2}|f^i|^2(x^1)^c,$$
$$N|u|^{p-2}|f^i|^2(x^1)^c \leq \delta|M^{-1}u|^p(x^1)^{\theta-d} + N|f^i|^p(x^1)^{\theta-d},$$
$$-pc|u|^{p-2}uf^i(x^1)^{c-1} \leq \delta|M^{-1}u|^p(x^1)^{\theta-d} + N|f^i|^p(x^1)^{\theta-d},$$
$$-pc|u|^{p-2}ua^{ij}u_{x^j}(x^1)^{c-1} \leq \varepsilon|u|^{p-2}|u_x|^2(x^1)^c + \frac{1}{4\varepsilon}(Kcp)^2|M^{-1}u|^p(x^1)^{\theta-d}.$$

Hence,

$$(\delta_0p(p-1)-\varepsilon-\delta)E\int_0^t \int_{\mathbb{R}^d_+} |u|^{p-2}|u_x|^2(x^1)^c\mathrm{d}x\mathrm{d}t$$
$$\leq \left(\frac{1}{4\varepsilon}+\delta\right)(Kcp)^2E\int_0^t \int_{\mathbb{R}^d_+} |M^{-1}u|^p(x^1)^{\theta-d}\mathrm{d}x\mathrm{d}t$$
$$+ NE\int_0^t \int_{\mathbb{R}^d_+} |f^i|^p(x^1)^{\theta-d}\mathrm{d}x\mathrm{d}t + NE\int_0^t \int_{\mathbb{R}^d_+} |g|^p(x^1)^{\theta-d}\mathrm{d}x\mathrm{d}t.$$

By corollary 6.2 in [9],

$$(1-c)^2p^{-2}\int_{\mathbb{R}^d_+} |M^{-1}u|^p(x^1)^{\theta-d} \leq \int_{\mathbb{R}^d_+} |u|^{p-2}|u_x|^2(x^1)^c\mathrm{d}x.$$

Now take $\varepsilon = \frac{1}{2}\delta_0p(p-1)$ and get

$$(1-c)^2p^{-2}(\delta_0p(p-1)-2\delta)E\int_0^t \int_{\mathbb{R}^d_+} |M^{-1}u|^p(x^1)^{\theta-d}\mathrm{d}x\mathrm{d}t$$
$$\leq \left(\frac{1}{\delta_0p(p-1)}+2\delta\right)(Kcp)^2E\int_0^t \int_{\mathbb{R}^d_+} |M^{-1}u|^p(x^1)^{\theta-d}\mathrm{d}x\mathrm{d}t$$

$$+NE\int_0^t\int_{\mathbb{R}^d_+}|f^i|^p(x^1)^{\theta-d}\mathrm{d}x\mathrm{d}t+NE\int_0^t\int_{\mathbb{R}^d_+}|g|^p(x^1)^{\theta-d}\mathrm{d}x\mathrm{d}t.$$

Notice that if (5.15) holds then for sufficiently small $\delta>0$,

$$(1-c)^2p^{-2}(\delta_0p(p-1)-2\delta)>\left(\frac{1}{\delta_0p(p-1)}+2\delta\right)(Kcp)^2.$$

Thus (5.16) follows in this case. If $\theta=d$ then by Lemma 2.12, Remark 2.10, Theorems 2.8 and 2.9 (remember that u has a compact support in $\mathbb{R}^d_+$),

$$\begin{aligned}\|M^{-1}u\|_{\mathbb{L}_{p,d}}\le N\|u_x\|_{\mathbb{L}_{p,d}}&=N\|u_x\|_{\mathbb{L}_p(\mathbb{R}^d)}\le N\|f^i\|_{\mathbb{L}_p(\mathbb{R}^d)}+N\|g\|_{\mathbb{L}_p(\mathbb{R}^d)}\\&=N\|f^i\|_{\mathbb{L}_{p,d}}+N\|g\|_{\mathbb{L}_p(\mathbb{R}^d)}.\qquad\square\end{aligned}$$

Lemma 5.8. *Let (2.4) hold with one of* (a)–(c) *in Lemma 5.6. Then there exists $\kappa_1>0$ such that if (2.19) holds then for any $f^i\in\mathbb{L}_{p,\theta}(T)$, $g\in\mathbb{L}_{p,\theta}(T)$, Eq. (4.1) with $f=0$ has a unique solution $u\in\mathfrak{H}^1_{p,\theta,0}(T)$ and*

$$\|u\|_{\mathfrak{H}^1_{p,\theta}(T)}\le N\|f^i\|_{\mathbb{L}_{p,\theta}(T)}+N\|g\|_{\mathbb{L}_{p,\theta}(T)}.\tag{5.17}$$

Proof. As usual we only show that there exists $\kappa_1>0$ such that estimate (5.17) holds given that a solution $u\in\mathfrak{H}^1_{p,\theta,0}(T)$ already exists and (2.19) holds. We also assume that u is sufficiently smooth and has compact support. Hence, (5.17) follows from Lemmas Lemmas 5.6 and 5.7 if $\theta=d$ or

$$\theta_0:=d-2+p-\frac{1}{\frac{Kp}{\delta_0(p-1)}-1}<\theta<d-2+p+\frac{1}{\frac{Kp}{\delta_0(p-1)}+1}=:\theta_1.$$

Consequently the lemma holds true in these cases, and also by a perturbation argument on θ (see Lemma 3.3 in [1]) one can easily show that estimate (5.17) holds if $\theta\in(d-\kappa_1,d+\kappa_1)$ for some $\kappa_1>0$. Thus we are done if $\theta_0<d$, and we only need to consider the case $d\le\theta\le\theta_0$. Fix $\theta_2\in(\theta_0,\theta_1)$, then the operator

$$\mathcal{R}:(f^i,g)\in(\mathbb{L}_{p,\theta}(T))^d\times\mathbb{L}_{p,\theta}(T)\to u\in M\mathbb{L}_{p,\theta}(T)$$

where $u\in\mathfrak{H}^1_{p,\theta,0}(T)$ is the solution of (4.1) with $f=0$, is bounded if $\theta=d$ or $\theta=\theta_2$. The arguments in the proof of Theorem 3.2 in [7] show that this operator is bounded for any $d\le\theta\le\theta_2$. This, with Lemma 5.6, finishes the proof. $\square$

Proof of Theorem 5.4. Take κ_1 from Lemma 5.8. We will show that there exists $\beta_0=\beta_0(\theta,d,p)\in(0,1)$ such that if $\beta\le\beta_0$ and (2.19) holds, then (5.9) holds true given that a solution $u\in\mathfrak{H}^1_{p,\theta}(T)$ already exists. We consider the case only when (b) holds. The other cases are treated similarly. Fix $x_0\in\mathbb{R}^d_+$ and denote $\sigma^{ik}_0(t,x)=\sigma^{ik}(t,x_0)$. Let $v\in\mathfrak{H}^1_{p,\theta}(T)$ be the solution of

$$\mathrm{d}v=(\Delta v+\bar{f})\mathrm{d}t+\nu^ku\mathrm{d}w^k_t,\quad v(0)=u_0.$$

where $\bar{f}=(\bar{b}^iu)_{x^i}+b^iu_{x^i}+cu+f$. Then by Theorem 3.3 in [10],

$$\begin{aligned}\|M^{-1}v\|_{\mathbb{H}^1_{p,\theta}(T)}&\le N\|M\bar{f}\|_{\mathbb{H}^{-1}_{p,\theta}(T)}+N\|\nu u\|_{\mathbb{L}_{p,\theta}(T)}+N\|u_0\|_{U^1_{p,\theta}}\\&\le N\beta\|M^{-1}u\|_{\mathbb{H}^1_{p,\theta}(T)}+N\|u_0\|_{U^1_{p,\theta}}.\end{aligned}\tag{5.18}$$

Note that $\bar{u} = u - v$ satisfies

$$d\bar{u} = D_i(a^{ij}\bar{u}_{x^j} + \bar{f}^i)dt + (\sigma_0^{ik}\bar{u}_{x^i} + \bar{g}^k)dw_t^k$$

where

$$\bar{f}^i = (a^{ij} - \delta^{ij})v_{x^i} + f^i, \qquad \bar{g}^k = g^k + \sigma^{ik}v_{x^i} + (\sigma^{ik} - \sigma_0^{ik})\bar{u}_{x^i}.$$

Since the coefficients σ_0^{ik} are independent of x, by Lemma 5.8 (also remember that $|(\sigma - \sigma_0)u_x| \leq \beta|u_x|$)

$$\|\bar{u}\|_{\mathbb{H}^1_{p,\theta}(T)} \leq N\|f^i\|_{\mathbb{L}_{p,\theta}(T)} + N\|v_x\|_{\mathbb{L}_{p,\theta}(T)} + N\beta\|\bar{u}\|_{\mathbb{L}_{p,\theta}(T)} + N\|g\|_{\mathbb{L}_{p,\theta}(T)}.$$

This and (5.18) finish the proof.

6. Proof of Theorem 2.17

As usual we assume that $\tau \equiv T$. First we establish an a priori estimate (2.13) given that a solution u already exists. Remember that the theorem is already known if $p = q$ ([3], Theorem 2.9). Repeat the proof of Theorem 2.9 in [3] using Theorems 2.5 and 5.2 instead of the corresponding versions (when $p = q$) in [3], and get for each $t \leq T$

$$\begin{aligned}\|u\|^q_{\mathfrak{H}^{\gamma+2,q}_{p,\theta}(\mathcal{O},t)} \leq{}& N\|u\|^q_{\mathbb{H}^{\gamma+1,q}_{p,\theta}(\mathcal{O},t)} + N\|\psi f\|^q_{\mathbb{H}^{\gamma,q}_{p,\theta}(\mathcal{O},T)}\\ &+ N\|g\|^q_{\mathbb{H}^{\gamma+1,q}_{p,\theta}(\mathcal{O},T)} + NE\|\psi^{2/q-1-\varepsilon_0}u_0\|^q_{H^{\gamma+2-2/q+\varepsilon_0}_{p,\theta}(\mathcal{O})}.\end{aligned}$$

Now we use the inequality

$$E\sup_{s\leq t}\|u(s)\|^q_{H^{\gamma+1}_{p,\theta}(\mathcal{O})} \leq N\|u\|^q_{\mathfrak{H}^{\gamma+2,p}_{p,\theta}(\mathcal{O},t)} \tag{6.1}$$

to get

$$\|u\|^q_{\mathbb{H}^{\gamma+1,q}_{p,\theta}(\mathcal{O},t)} \leq N\int_0^t \|u\|^q_{\mathfrak{H}^{\gamma+2,q}_{p,\theta}(\mathcal{O},s)}dt. \tag{6.2}$$

Inequality (6.1) is from (2.21) of [13] if $p = q$, and it follows from inequalities (4.5) and (4.18) of [6] if $q > p$. Thus Gronwall's inequality lead to (2.13).

The a priori estimate combined with the method of continuity shows that it only remains to prove solvability of the equation

$$du = (\Delta u + f)dt + g^k dw_t^k, \quad u(0,\cdot) = u_0. \tag{6.3}$$

One can approximate u_0 with smooth functions with compact support (see [12]). Then considering the difference $u - u_0$ we see that we may assume that $u_0 = 0$. We can also approximate f and g with functions in $\mathbb{H}^{\gamma,q}_{q,\theta}(\mathcal{O},T)$ and $\mathbb{H}^{\gamma+1,q}_{q,\theta}(\mathcal{O},T)$, respectively. Thus due to (2.13) we may assume that $f \in \mathbb{H}^{\gamma,q}_{q,\theta}(\mathcal{O},T)$ and $g \in \mathbb{H}^{\gamma+1,q}_{q,\theta}(\mathcal{O},T)$. In this case by Theorem 2.9 of [3], there exists a solution $u \in \mathfrak{H}^{\gamma+2,q}_{q,\theta}(\mathcal{O},T)$. Since $p \leq q$, it follows that $u \in \mathfrak{H}^{\gamma+2,q}_{p,\theta}(\mathcal{O},T)$. The theorem is proved.

7. Proof of Theorem 2.15

The theorem is proved in [2] if $p = q$. To prove Theorem 2.15 it suffices to repeat the proof of Theorem 2.7 in [2] using Theorems 2.3 and 5.3 instead of the corresponding versions (when $p = q$) in [2].

8. Proof of Theorems 2.19–2.21

On the basis of Theorems 2.8, 2.9 and 5.4, it suffices to repeat the arguments in the proof of Theorem 2.4 in [1], where the theorems are proved if $\theta \approx d$ and $\sigma = 0$.

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